EFFECT OF FREQUENCY OFFSET ON BER OF OFDM AND SINGLE CARRIER SYSTEMS

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Abstract— **Performance of both orthogonal frequency division multiplexing (OFDM) and single carrier (SC) systems suffers from a carrier frequency offset (CFO) due to Doppler effect and the carrier instability between the transmitter and the receiver. We investigate the bit error rate (BER) performance degradation of OFDM and SC systems due to the frequency offset in an additive white Gaussian noise (AWGN) channel as well as multipath Rayleigh fading channels. We consider three effects to the BER degradation, i.e., phase shift, useful power decrease and intercarrier interference (ICI). We also derive the approximate expressions of BER under binary phase shift keying (BPSK) and quaternary phase shift keying (QPSK) for both OFDM and SC systems in the presence of CFO. In general, SC is more robust to CFO in the AWGN channel than OFDM in terms of BER, while both of them suffer similarly from CFO in the multipath Rayleigh fading channels assuming the the same CFO.**

Index Terms— **OFDM, single carrier, channel frequency offset**

I. Introduction

Orthogonal frequency division multiplexing [1], a spectrally efficient form of frequency division multiplexing (FDM), divides its allocated channel spectrum into several parallel sub-channels. OFDM is inherently robust against frequency selective fading, since each sub-channel occupies a relatively narrow band, where the channel frequency characteristic is nearly flat. A corresponding system proposed for high speed data transmission is a SC system with the cyclic prefix (CP) inherited from OFDM[3][7]. SC systems with cyclic prefix has a simple equalization method in the frequency domain.

The OFDM and SC signals are usually transmitted after its baseband signal modulates a radio carrier frequency, e.g., 5GHz for IEEE 802.11a. Although this frequency is known to the receiver, the tolerance of radio frequency (RF) components is usually so large that there will be a non-negligible frequency deviation. Another source of frequency offset is the Doppler shift caused by the relative speed between the corresponding transmitter and receiver or the motion of other objects around transceivers. There are three problems affected by this frequency offset: phase shift, effective power decrease and ICI. All of them will degrade the performance of OFDM and SC systems in terms of BER. Previous work [2][5] concludes that OFDM systems are much more sensitive to CFO than the SC systems. However, that conclusion is based on a fixed symbol rate. In the case of fixed symbol rate the OFDM systems have much faster bit rate determined by the effective number of subcarriers. Thus, it is not a fair comparison. Furthermore, almost all the studies of BER degradation from CFO in OFDM ignore the phase rotation in each subcarrier. In practice this phase rotation can not be easily compensated when the CFO is unknown. Consequently, the effect of the phase rotation must be taken into account in evaluating the BER. Sathananthan [4] proposes a precise technique for calculating the effect of the CFO on the BER in an OFDM system on the basis of an infinite series expression for the error function. However, the author only considers the BER in the AWGN channel and ignores the phase rotation.

In this paper, we derive the approximate BERs for the OFDM and SC systems in the presence of CFO under the AWGN channel. For OFDM systems, the ICI is approximated as Gaussian noise. For multipath Rayleigh fading channels, we employ simulations to obtain the BER performance of SC systems since analytical results are difficult to derive.

The rest of the paper is organized as follows. In Section II we will describe the baseband OFDM and SC system models in the presence of frequency offset and make some assumptions. BERs of OFDM and SC in the AWGN and multipath Rayleigh fading channels are presented in Section III and IV, respectively. Finally, we draw some conclusions in Section V.

II. System Model of OFDM and SC

A. System Model of OFDM

The OFDM system model over a multipath slowly time varying fading channel with CFO can be described as

$$y_1(k) = e^{j2\pi\frac{k\epsilon}{M}} \sum_{l=0}^{L-1} h_l x_1(k-l) + n_1(k), \quad 0 \le k \le M-1, \tag{1}$$

where h_l's ($0 \le l \le L-1$) are i.i.d. complex-valued Rayleigh fading random variables, and $n_1(k)$'s ($0 \le k \le M-1$) are independent complex-valued Gaussian random variables with zero mean and variance σ_n^2 for both real and imaginary components. $x_1(0), \cdots, x_1(M-1)$[1] are time domain transmitted symbols. L is the length of the time-domain channel

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[1]In this paper we use subscript 1 to denote OFDM related variables and subscript 2 to denote SC related variables.

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impulse response (CIR). It should be noted that this model can be employed in the AWGN channel by setting $h_0 = 1$ and $h_i = 0, i \geq 1$. ϵ is the channel frequency offset which is normalized by the subcarrier spacing which is roughly $\frac{f_{OFDM}}{M}$, where f_{OFDM} is the total occupied bandwidth by OFDM. We assume the integer part of the frequency offset has been estimated and compensated so that the remaining fractional frequency offset is within one subcarrier spacing (i.e., $|\epsilon| \leq \frac{1}{2}$).

If we add a cyclic prefix in each OFDM data frame, with its length chosen to be longer than L, there will be no inter-frame interference (IFI) among the OFDM frames. Thus we need to consider only one OFDM frame with M sub-carriers to analyze the system performance. After discarding the cyclic prefix and performing an FFT at the receiver, we can obtain the received data frame in the frequency domain:

$$Y_1(m) = \frac{\sin \pi\epsilon}{M \sin \frac{\pi\epsilon}{M}} X_1(m)H(m)e^{j\pi\frac{(M-1)\epsilon}{M}} + ICI_1(m) + N_1(m), \quad (2)$$

where $H(m)$ is the frequency response of the channel at sub-carrier m and $N(m)$ is the frequency domain Gaussian noise variable at sub-carrier m. The noteworthy term in (2) is $ICI_1(m)$, which is defined by

$$ICI_1(m) = \frac{1}{M}\sum_{k=0}^{M-1}\sum_{n\neq m} X_1(n)H(n)e^{j2\pi\frac{k(n-m+\epsilon)}{M}}, \; n \neq m, \quad (3)$$

which is nonzero if $\epsilon \neq 0$.

Equation (2) shows that the frequency offset degrades the amplitude of the received signal in each suncarrier and introduce inter-carrier interference (ICI). In addition, a common phase shift $\pi\frac{(M-1)\epsilon}{M}$ is introduced to the received signal.

B. System Model of SC

Assuming similar system parameters to those of OFDM, especially the same frequency offset ϵ, a SC system can be described as

$$y_2(k) = e^{j2\pi\frac{k\epsilon}{M}} \sum_{l=0}^{L-1} h_l x_2(k-l) + n_2(k), \qquad 0 \leq k \leq M-1, \quad (4)$$

where $x_2(0), \cdots, x_2(M-1)$ are MPSK or MQAM modulated signals without the IFFT operation. ϵ is the channel frequency offset which is normalized by $\frac{f_{SC}}{M}$, where f_{SC} is the total occupied bandwidth by the SC system. Under the definitions of ϵ for OFDM and SC, the same actual frequency offset leads to the same ϵ when $f_{SC} = f_{OFDM}$. decision statistic of demodulating $x_2(i)$ only depends on $x_2(i)$ itself and the additive noise. On the contrary, the decision statistic of demodulating $X_1(i)$ depends on $X_1(0), \cdots, X_1(M-1)$ and the additive noise due to the FFT operation. This intuitive observation reveals that OFDM is more sensitive to CFO than SC in the AWGN channel, which can be verified in Section III. However, the difference of the CFO sensitivity of OFDM and SC in the multipath rayleigh fading channels is not obvious.

C. Simulation Model

The simulation parameters for OFDM are as follows: the entire channel bandwidth 800kHz is divided into $M = 64$ sub-carriers (or tones), hence, the subcarrier spacing is 12.5kHz. To make the tones orthogonal to each other, the frame symbol duration is 80 μs. An additional 20 μs long cyclic prefix is added to provide protection from IFI and ICI due to channel delay spread. Thus, the total OFDM frame length is $T_s = 100$ μs and the subchannel symbol rate is 10 kbaud. The modulation schemes used in the system are BPSK and QPSK. The simulated system thus transmits uncoded data at 0.64 Mbits/s for BPSK and 1.28 Mbits/s for QPSK. The maximum Doppler frequency f_d is chosen to be 100Hz, which makes $f_d T_s = 0.01$. The channel impulse response used in the simulation is

$$h_l = \frac{1}{C}\sum_{k=0}^{7} e^{-k/2}\alpha_k \delta_{l,k},$$

where $C = \sqrt{\sum_{k=0}^{7} e^{-k}}$ is the normalization constant and $\alpha_k, 0 \leq k \leq 7$ are i.i.d. complex-valued Gaussian distributed random variables with zero mean and unit variance, which vary in time according to the Doppler frequency. The amplitude of α_k is Rayleigh distributed. This is a conventional exponential decay multipath fading channel model.

Similar parameters are adopted for the SC system such as 64 symbols in one block, and a 16-symbol additional cyclic prefix is added in the front of each block. Both OFDM and SC are operated in the same frequency band with the same carrier frequency.

III. BER IN THE AWGN CHANNEL

We first study the BER performance in the presence of CFO for OFDM in the AWGN channel. Since the CFO is not known at the receiver, the common phase shift $\pi\frac{(M-1)\epsilon}{M}$ in all subcarriers is unknown and causes degradation of the BER. We first give the following lemma that will be useful to derive the approximate BER in the sequel.

Lemma 1: Suppose $X_1(0), \cdots, X_1(M-1)$ are independent with zero mean and variance $E\{|X_1(n)|^2\} = \sigma_X^2$ and $E\{|H(n)|^2\} = E\left\{\sum_{l=0}^{L-1}|h_l|^2\right\} = E_{\underline{h}}$ for all subcarriers noted by $n, 0 \leq n \leq M-1$. Define

$$I_{m,n} = \frac{1}{M}\sum_{k=0}^{M-1} X_1(n)H(n)e^{j2\pi\frac{k(n-m+\epsilon)}{M}}. \quad (5)$$

Then, the following equality holds

$$E\left\{\left(\sum_{n=0}^{M-1}|I_{m,n}|\right)^2\right\} = E\left\{\sum_{n=0}^{M-1}|I_{m,n}|^2\right\} = \sigma_X^2 E_{\underline{h}} \quad (6)$$

for each subcarrier m.

Proof: The proof is straightforward and we omit it due to lack of space. ∎

For the AWGN channel, $E_h = 1$ and

$$\begin{aligned} E\left\{|ICI_1(m)|^2\right\} &= E\left\{\sum_{n=0,n\neq m}^{M-1} |I_{m,n}|^2\right\} \\ &= \sigma_X^2\left(1 - E\left\{|I_{m,m}|^2\right\}\right) \\ &= \sigma_X^2\left(1 - \frac{\sin^2 \pi\epsilon}{M^2 \sin^2 \frac{\pi\epsilon}{M}}\right). \end{aligned} \quad (7)$$

If the CFO is small enough, i.e., $\epsilon \ll 1$, (7) becomes

$$E\left\{|ICI_1(m)|^2\right\} \approx \frac{\sigma_X^2 (\pi\epsilon)^2}{3}, \quad (8)$$

which was initially derived in [5]. However, Eq. (7) becomes much more accurate than Eq. (8) when ϵ increases. As usual $ICI_1(m)$ can be approximated as a Gaussian distributed random variable because it is a supperposition of independent identical distributed (i.i.d.) random variables. Although a more accurate BER has been derived in [4], it is only suitable for the AWGN channel. Our Gaussian approximation, on the other hand, can also be employed in multipath fading channels. By this Gaussian approximation the analytical BER with BPSK can be obtained as

$$BER_1 \approx Q\left(\sqrt{\frac{\frac{\sigma_X^2 \sin^2 \pi\epsilon}{M^2 \sin^2 \frac{\pi\epsilon}{M}} \cos^2 \frac{\pi\epsilon(M-1)}{M}}{\sigma_X^2\left(1 - \frac{\sin^2 \pi\epsilon}{M^2 \sin^2 \frac{\pi\epsilon}{M}}\right) + \sigma_n^2}}\right), \quad (9)$$

and the analytical BER with QPSK can be expressed as

$$\begin{aligned} BER_2 \approx\; & \frac{1}{2} Q\left(\sqrt{\frac{\frac{\sigma_X^2 \sin^2 \pi\epsilon}{M^2 \sin^2 \frac{\pi\epsilon}{M}} \cos^2\left(\frac{\pi}{4} + \frac{\pi\epsilon(M-1)}{M}\right)}{\frac{\sigma_X^2}{2}\left(1 - \frac{\sin^2 \pi\epsilon}{M^2 \sin^2 \frac{\pi\epsilon}{M}}\right) + \sigma_n^2}}\right) \\ + & \frac{1}{2} Q\left(\sqrt{\frac{\frac{\sigma_X^2 \sin^2 \pi\epsilon}{M^2 \sin^2 \frac{\pi\epsilon}{M}} \sin^2\left(\frac{\pi}{4} + \frac{\pi\epsilon(M-1)}{M}\right)}{\frac{\sigma_X^2}{2}\left(1 - \frac{\sin^2 \pi\epsilon}{M^2 \sin^2 \frac{\pi\epsilon}{M}}\right) + \sigma_n^2}}\right) \end{aligned} \quad (10)$$

where it is obtained by computing the signal to interference plus noise ratios (SINR) for the real and imaginary components.

For a SC system operating in the AWGN channel, channel equalization is not necessary. Thus, there is no interference between symbols in one block. However, the CFO causes the error probability of each symbol to be different. Therefore, the exact BER of SC with BPSK is

$$BER_3 = \frac{1}{M} \sum_{k=0}^{M-1} Q\left(\sqrt{\frac{\cos^2 \frac{2k\pi\epsilon}{M}}{\sigma_n^2}}\right). \quad (11)$$

When $M \gg 1$, the above BER can be approximted by an integral

$$BER_3 \approx \int_0^1 Q\left(\sqrt{\frac{\cos^2(2\pi\epsilon x)}{\sigma_n^2}}\right) dx. \quad (12)$$

The corresponding BER of SC with QPSK can be obtained using a similar technique:

$$\begin{aligned} BER_4 = & \frac{1}{2M} \sum_{k=0}^{M-1} [Q\left(\sqrt{\frac{\cos^2\left(\frac{\pi}{4} + \frac{2k\pi\epsilon}{M}\right)}{\sigma_n^2}}\right) \\ & + Q\left(\sqrt{\frac{\sin^2\left(\frac{\pi}{4} + \frac{2k\pi\epsilon}{M}\right)}{\sigma_n^2}}\right)]. \end{aligned} \quad (13)$$

When $M \gg 1$, the above BER can be approximted by:

$$\begin{aligned} BER_4 \approx & \frac{1}{2} \int_0^1 [Q\left(\sqrt{\frac{\cos^2\left(\frac{\pi}{4} + 2\pi\epsilon x\right)}{\sigma_n^2}}\right) \\ & + Q\left(\sqrt{\frac{\sin^2\left(\frac{\pi}{4} + 2\pi\epsilon x\right)}{\sigma_n^2}}\right)] dx. \end{aligned} \quad (14)$$

It should be noted that the exact and analytical BERs for SC hold only when $|\epsilon| < \frac{1}{4}$ and $|\epsilon| < \frac{1}{8}$ for BPSK and QPSK respectively. This restriction comes from the fact that the phase shift is so large that it rotates the desired signal out of the correct decision region. The corresponding restrictions for OFDM are $|\epsilon| < \frac{1}{2}$ and $|\epsilon| < \frac{1}{4}$ for BPSK and QPSK respectively. From the the point of view of CFO tolerance, OFDM has a larger permissible range .

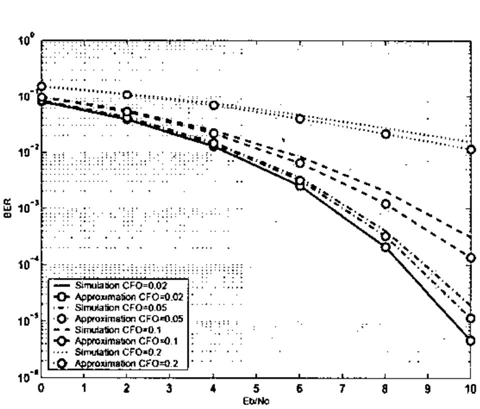

Fig. 1. Simulated and analytical BERs of OFDM in the AWGN channel with BPSK modulation under different CFOs

Figs. 1-4 illustrate the BERs for OFDM and SC in the AWGN channel with BPSK and QPSK modulations under different CFOs. Both simulation and analytical BER curves are displayed in these figures. Although the Gaussian approximation for ICI in OFDM is very simple, it leads to BERs with acceptable precision, especially for QPSK modulation. For BPSK modulation the analytical BERs for OFDM are a bit optimistic. The analytic and simulation BERs for SC match very well except for QPSK with $\epsilon = 0.2$. The reason is fairly simple, i.e., the CFO exceeds $\frac{1}{8}$, which violates the condition with which the analytical result is meaningless.

Comparing the corresponding simulated BER curves in Figs. 1-3, we found that SC-BPSK is less sensitive to CFO

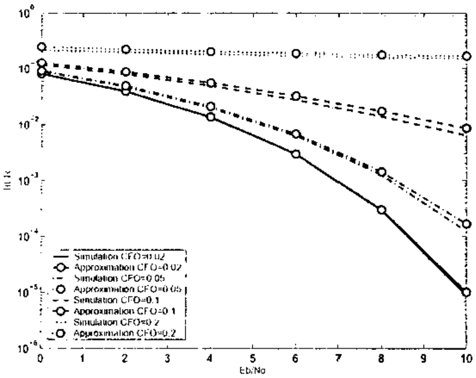

Fig. 2. Simulated and analytical BERs of OFDM in the AWGN channel with QPSK modulation under different CFOs

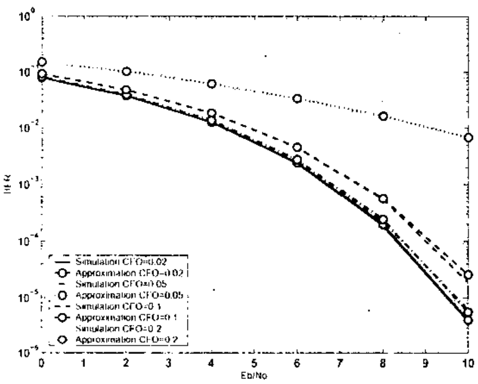

Fig. 3. Simulated and analytical BERs of SC in the AWGN channel with BPSK modulation under different CFOs

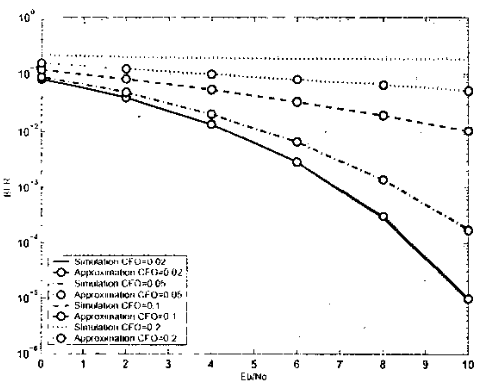

Fig. 4. Simulated and analytical BERs of SC in the AWGN channel with QPSK modulation under different CFOs

than OFDM-BPSK in terms of BER. However, SC-QPSK and OFDM-QPSK have similar BER performance for the same CFOs. It is quite different from the previous conclusions made in [2] and [5]. Remember all of the above results are obtained under the assumption that both OFDM and SC are subject to the same CFO.

IV. BER in Multipath Rayleigh Fading Channels

To calculate the BERs of OFDM in the multipath Rayleigh fading channels the simple Gaussian approximation of ICI does not give a good BER approximation. Instead, we find an analytical BER for OFDM in the multipath fading channel with BPSK modulation as

$$BER_5 \approx \int_0^\infty Q\left(\sqrt{\frac{\frac{\sigma_X^2 \sin^2 \pi\epsilon}{M^2 \sin^2 \frac{\pi\epsilon}{M}} \cos^2 \frac{\pi\epsilon(M-1)}{M} r^2}{\sigma_X^2\left(1-0.93\frac{\sin^2 \pi\epsilon}{M^2 \sin^2 \frac{\pi\epsilon}{M}}\right) r^2+\sigma_n^2}}\right) 2re^{-r^2}dr \qquad (15)$$

An analytical BER for OFDM in the multipath fading channel with QPSK modulation is expressed in Eq. 16.

The above analytical BERs are obtained by first assuming the channel frequency responses for all subcarriers are the same, i.e, the channel is frequency flat, and then taking expectation with respect to the channel frequency responses. The two constants 0.93 and 0.97 are the adjustments for the frequency selective channel. Although the derivation of the analytical BERs is empirical, they match the simulation results very well (See Figs. 5 and 6). In a broad range of CFO, i.e., $\epsilon = 0.02 - 0.2$, the analytical BER curves are very close to the corresponding simulation curves both for BPSK and QPSK.

It is much more difficult to analyze the BERs of SC in the multipath fading channel even without CFO. The BERs are highly dependent on the different equalization methods [8]. So, we can only resort to numerical results (Figs. 7 and 8)to demonstrate the effect of CFO on the BERs of SC. We adopt frequency domain minimum mean square error (FD-MMSE) as the equalization method because of its good performance for the whole SNR range we are interested in and its low complexity [8]. Comparing Figs. 5 with 7 for BPSK and 6 with 8 for QPSK, we find that when $\epsilon < 0.1$ the BERs of SC are much smaller than the corresponding BERs of OFDM. This is largely because of the lower BERs of SC with FD-MMSE in the absence of CFO [8]. However, when $\epsilon > 0.1$, OFDM and SC have comparable performance.

V. Conclusion

In this paper we compared the sensitivity to CFO both for OFDM and SC. In particular, we investigated the bit error rate (BER) performance degradation of OFDM and SC systems due to the frequency offset in the AWGN channel and the multipath Rayleigh fading channel. We considered three effects to the BER degradation. i.e., phase shift, useful power decrease and ICI. We also derive the approximate expressions of BER under BPSK and QPSK for both OFDM and SC systems in the presence of CFO. Those simple analytical BERs are very close to the simulated BERs. Surprisingly, it is shown

$$BER_6 \approx \frac{1}{2}\int_0^{\infty}\left[Q\left(\sqrt{\frac{\frac{\sigma_X^2 \sin^2 \pi\epsilon}{M^2 \sin^2 \frac{\pi\epsilon}{M}} \cos^2\left(\frac{\pi}{4}+\frac{\pi\epsilon(M-1)}{M}\right) r^2}{\frac{\sigma_X^2}{2}\left(1-0.97\frac{\sin^2 \pi\epsilon}{M^2 \sin^2 \frac{\pi\epsilon}{M}}\right) r^2+\sigma_n^2}}\right)+Q\left(\sqrt{\frac{\frac{\sigma_X^2 \sin^2 \pi\epsilon}{M^2 \sin^2 \frac{\pi\epsilon}{M}} \sin^2\left(\frac{\pi}{4}+\frac{\pi\epsilon(M-1)}{M}\right) r^2}{\frac{\sigma_X^2}{2}\left(1-0.97\frac{\sin^2 \pi\epsilon}{M^2 \sin^2 \frac{\pi\epsilon}{M}}\right) r^2+\sigma_n^2}}\right)\right] 2r e^{-r^2} dr \quad (16)$$

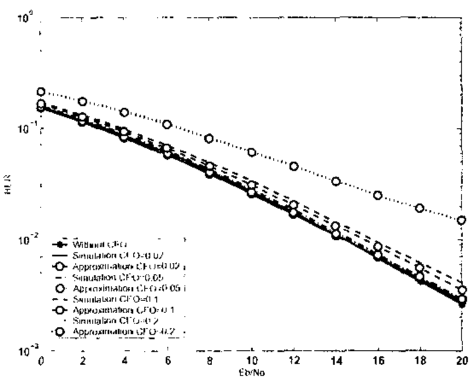

Fig. 5. Simulated and analytical BERs of OFDM in the multipath fading channel with BPSK modulation under different CFOs

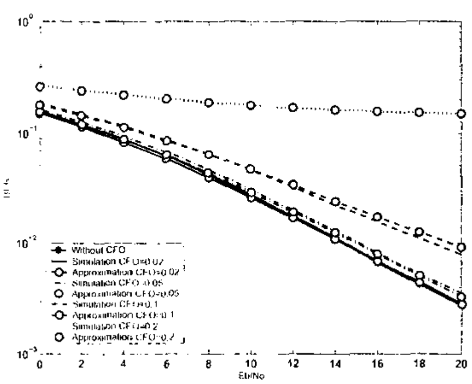

Fig. 6. Simulated and analytical BERs of OFDM in the multipath fading channel channel with QPSK modulation under different CFOs

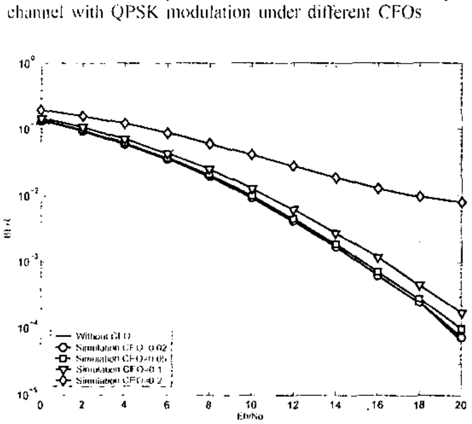

Fig. 7. Simulated BERs of SC in the multipath fading channel channel with BPSK modulation under different CFOs

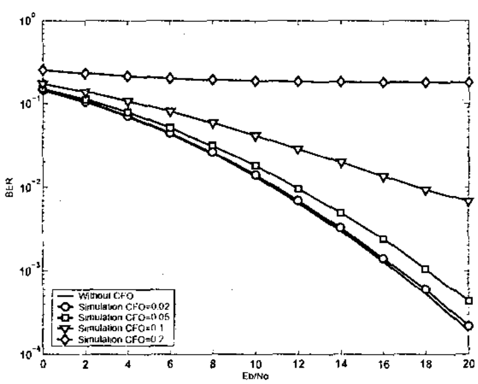

Fig. 8. Simulated BERs of SC in the multipath fading channel channel with QPSK modulation under different CFOs

that OFDM can tolerate a larger variation of CFO than SC. It is also demonstrated that SC-BPSK is less sensitive to CFO than OFDM-BPSK while SC-QPSK and OFDM-QPSK have similar BER performance in the AWGN channel. Furthermore, both of OFDM and SC suffer similarly from CFO in multipath Rayleigh fading channels assuming the the same CFO.

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