A Diffusion Approximation Analysis of an ATM Statistical Multiplexer with Multiple Types of Traffic Part I: Equilibrium State Solutions

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Abstract

We introduce a multi-dimensional diffusion model to characterize the "on-off" sources behavior in an ATM statistical multiplexer, where multiple types of traffic are concentrated. Under a reasonable set of assumptions, this diffusion process can then be approximated by a multi-dimensional Ornstein-Uhlenbeck process, which is a Gaussian Markov process. The packet arrival process is shown to be a Gaussian (but not Markov) process, and this process determines the statistical behavior of the buffer content. We then derive an expression for the joint probability distribution of the buffer content and "on-off" sources in the equilibrium state. The final solution form is given in terms of the eigenfunctions of Weber's equation. Some numerical case is compared with the solution method developed by Kosten [1984]. In a companion paper (Ren and Kobayashi [1992b]), we shall derive the time-dependent solution of the diffusion approximation model.

1 Introduction

In a future B-ISDN(broadband integrated services digital network), multiple types of information services will be provided by means of fast packet switching with statistical multiplexers. The traffic into a statistical multiplexer is a superposition of packet streams from many sources of differing types. The instantaneous packet arrival rate generally depends on the number of sources in "burst" (or "on") states and thus fluctuates with high variability. Therefore, the traffic is usually far from Poisson or any type of renewal process: there is a positive serial dependence between successive packet arrivals, and this dependency is a major cause of congestion in the multiplexer queue and often leads to surprisingly large packet delays or packet loss under heavy traffic conditions.

There have been a number of studies that report analytic models of statistical multiplexing. Hashida and Fujiki [1973], Kosten [1974,1984], Anick, Mitra and Sondhi [1982] formulate the problem as a Markovian system by assuming exponential distributions for both burst and silence periods. They assume that each burst generates packets at a constant rate, and arrival packets into the multiplexer output buffer is approximated by a fluid flow. Anick et al. and Kosten report comprehensive studies of this fluid model by calculating the eigenvalues of the resulting matrix equation that governs the underlying Markov process. Stern and Elwalid [1991] discussed solutions of the fluid model when the sources are Markov modulated.

All of the above studies, however, deal with the equilibrium-state solutions of the fluid model assuming a single type of traffic. Kosten [1984] and Kobayashi [1990] discuss the equilibrium solutions for multiple types of traffic. More recently, the authors (see Ren and Kobayashi [1992a], Kobayashi and Ren [1992]) obtained theoretical results on transient (or time-dependent) solutions of the statistical multiplexer model for multiple types of traffic. The results require, however, numerical inversions of the Laplace transforms, thus their practical applications await a further investigation of efficient computational algorithms and approximations.

In this paper we develop a *diffusion approximation* for the statistical multiplexer model and obtain computationally more feasible solutions than the results mentioned above. The idea of approximating a discrete-state process (e.g. a random walk) by a diffusion process with continuous path was discussed by Feller [1966] and others. Cox and Miller [1965] discussed applications of the diffusion process approach to congestion theory. The procedure of using a diffusion process to study a queueing system – whether it be a continuous-time system or a discrete-time system– can be useful because mathematical methods associated with the continuum very often lend themselves more easily to analytical treatment than those associated with discrete coordinate axes. See also Kleinrock [1976] and Kobayashi [1983] for expository treatments on the diffusion approximation method.

The diffusion processes we primarily deal with in the present and companion papers are an Ornstein-Uhlenbeck process (see e.g. Feller [1966], Cox and Miller [1965]) and its variants. An application of the Ornstein-Uhlenbeck (O-U) process to a communication network was discussed by

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Kobayashi, Onozato and Huynh [1977] and Kobayashi [1983] in their performance analysis of the ALOHA random access scheme. Kobayashi [1990] also discusses the O-U process to characterize the multiple on-off source models of a statistical multiplexer. More recently, Simonian [1991], Simonian and Virtamo [1991] discuss applications of an O-U process to the analysis of statistical multiplexer. We should also note that Knessl and Morrison [1991] discuss the heavy-traffic analysis of the model of Anick et al [1982], resulting in an O-U process representation.

In the present paper, we will extend the earlier results and derive a diffusion approximation for multiple types of traffic, and compute the equilibrium state distribution. In the companion paper (Ren and Kobayashi [1992b]), we shall discuss the transient analysis.

2 Derivation of a Diffusion Approximation Model

Let there be N_k sources of type k, where $k = 1, 2, \cdots, K$, and let $J_k(t)$ denote the number of type k sources in "on" (or "burst") state: the remaining $N_k - J_k(t)$ sources are "off" (or "silent"). We assume that successive "on" and "off" periods of each source form an alternating renewal process. For mathematical simplicity, we further assume that the "off" and "on" periods of type k sources are both exponentially distributed with parameters α_k and β_k, respectively:

$$\alpha_k^{-1} = \text{The mean silence period of a type } k \text{ source,} \quad (1)$$

$$\beta_k^{-1} = \text{The mean burst period of a type } k \text{ source.} \quad (2)$$

Let $\mathbf{j}$ be a vector defined by

$$\mathbf{j} = [j_1, j_2, \cdots, j_K], \quad (3)$$

where j_k is an integer that $J_k(t)$ can take on, $0 \le j_k \le N_k$.

Let R_k [packets/sec.] be the rate with which a type k source generates packets during its burst period. The aggregate packet arrival rate is therefore

$$R(t) = \sum_{k=1}^{K} R_k J_k(t). \quad (4)$$

Let C [packets/sec.] denote the link capacity of a multiplexer output, and $Q(t)$ be the queue size in the associated output buffer. Although $Q(t)$ is an integer-valued function, we approximate it by a time-continuous function as remarked earlier. Then we can relate the buffer queue size process $Q(t)$ to the packet arrival process $R(t)$ by

$$\frac{dQ(t)}{dt} = \begin{cases} R(t) - C, & \text{if } Q(t) > 0 \text{ or } R(t) > C, \\ 0, & \text{otherwise.} \end{cases} \quad (5)$$

Although neither $R(t)$ nor $Q(t)$ is a Markov process, the multivariate process $(\mathbf{J}(t), Q(t)) = (J_k(t), 1 \le k \le K; Q(t))$ is a Markov process. We define the probability distribution function

$$P(\mathbf{j}, x, t) = Prob[J_k(t) = j_k, 1 \le k \le K; \text{and } Q(t) \le x]. \quad (6)$$

Then we obtain the following partial differential equation that governs the process $(\mathbf{J}(t), Q(t))$

$$\begin{aligned} &\frac{\partial P(\mathbf{j}, x, t)}{\partial t} + \left(\sum_{k=1}^{K} R_k j_k - C\right)\frac{\partial}{\partial x} P(\mathbf{j}, x, t) \\ = &- \sum_{k=1}^{K} [(N_k - j_k)\alpha_k + j_k \beta_k] P(\mathbf{j}, x, t) \\ &+ \sum_{k=1}^{K} (N_k - j_k + 1)\alpha_k P(\mathbf{j} - \mathbf{1}_k, x, t) \\ &+ \sum_{k=1}^{K} (j_k + 1)\beta_k P(\mathbf{j} + \mathbf{1}_k, x, t), \end{aligned} \quad (7)$$

where $\mathbf{1}_k$ is a vector that has unity in its k-th entry and is zero elsewhere. If we consider the limit case, i.e., $x \to \infty$, Eq.(7) reduces to the system equation of a multi-dimensional birth-and-death process, yielding the following simple product form solution for the time-dependent solution

$$P(\mathbf{j}, t) \stackrel{\text{def}}{=} \lim_{x \to \infty} P(\mathbf{j}, x, t) = \prod_{k=1}^{K} P(j_k, t), \quad (8)$$

where

$$P(j_k, t) = \binom{N_k}{j_k} q_k^{j_k}(t)\{1 - q_k(t)\}^{N_k - j_k}, \quad (9)$$

and

$$q_k(t) = \frac{\alpha_k}{\alpha_k + \beta_k}\{1 - e^{-(\alpha_k + \beta_k)t}\}. \quad (10)$$

The expression of Eq.(9) is obtained by assuming that all the sources are initially off, i.e., $J_k(0) = 0$. A generalization to arbitrary initial condition is straightforward (see Kobayashi and Ren [1992]).

As is well known, the binomial distribution approaches a Gaussian distribution with mean $N_k q_k(t)$ and variance $N_k q_k(t)(1 - q_k(t))$ as N_k becomes large. This observation leads us to approximate the jump process $J_k(t)$ by a continuous process $Y_k(t)$. Then we define the corresponding probability distribution of the approximated multivariate process $(\mathbf{Y}(t), Q(t))$ by

$$f(\mathbf{y}, x, t)d\mathbf{y} = Prob[y_k \le Y_k(t) \le y_k + dy_k, 1 \le k \le K; \text{and } Q(t) \le x], \quad (11)$$

which is a density function with respect to the variables $Y_k(t)$, $(k = 1, 2, \cdots, K)$, but is a distribution function with respect to $Q(t)$. Then by applying the Taylor series expansions to the terms in Eq.(7) and retaining the first and the second order terms, we obtain the following equation

$$\frac{\partial f(\mathbf{y},x,t)}{\partial t} + (\sum_{k=1}^{K} R_k y_k - C)\frac{\partial f(\mathbf{y},x,t)}{\partial x}$$
$$= -\sum_{k=1}^{K}[(N_k - y_k)\alpha_k + y_k\beta_k]f(\mathbf{y},x,t)$$
$$+\sum_{k=1}^{K}\left\{(N_k - y_k + 1)\alpha_k[f(\mathbf{y},x,t) - \frac{\partial f(\mathbf{y},x,t)}{\partial y_k}]\right.$$
$$\left. + \frac{(N_k - y_k)\alpha_k}{2}\frac{\partial^2 f(\mathbf{y},x,t)}{\partial y_k^2}\right\}$$
$$+\sum_{k=1}^{K}\left\{(y_k + 1)\beta_k[f(\mathbf{y},x,t) + \frac{\partial f(\mathbf{y},x,t)}{\partial y_k}]\right.$$
$$\left. + \frac{y_k\beta_k}{2}\frac{\partial^2 f(\mathbf{y},x,t)}{\partial y_k^2}\right\}. \quad (12)$$

By a simple rearrangement, we can write

$$\frac{\partial f}{\partial t} + (\sum_{k=1}^{K} R_k y_k - C)\frac{\partial f}{\partial x}$$
$$= -\sum_{k=1}^{K}\frac{\partial}{\partial y_k}[m_k(y_k)f] + \sum_{k=1}^{K}\frac{1}{2}\frac{\partial^2}{\partial y_k^2}[v_k(y_k)f], \quad (13)$$

where we write f instead of $f(\mathbf{y},x,t)$, and

$$m_k(y_k) = N_k\alpha_k - (\alpha_k + \beta_k)y_k, \quad (14)$$

$$v_k(y_k) = N_k\alpha_k - (\alpha_k - \beta_k)y_k. \quad (15)$$

If we take the limit $x \to \infty$, then the marginal distribution function $f(\mathbf{y},+\infty,t) \stackrel{\text{def}}{=} f(\mathbf{y},t)$ satisfies

$$\frac{\partial f(\mathbf{y},t)}{\partial t} = \mathbf{L}\cdot f(\mathbf{y},t) \quad (16)$$

where $\mathbf{L}$ is a linear operator defined by

$$\mathbf{L}\cdot f = \sum_{k=1}^{K}\left\{-\frac{\partial}{\partial y_k}[m_k(y_k)f] + \frac{1}{2}\frac{\partial^2}{\partial y_k^2}[v_k(y_k)f]\right\} \quad (17)$$

Equation (16) is the *forward* diffusion equation (see e.g., Feller [1966]) of the Markov process $\mathbf{Y}(t)$, which is a diffusion approximation of the process $\mathbf{J}(t)$. The operator $\mathbf{L}$ is also known as the infinitesimal generator of the Markov process $\mathbf{Y}(t)$. The adjoint operator (see e.g. Wentzell [1981]) $\mathbf{L}^*$ can be easily found to be

$$\mathbf{L}^*\cdot f = \sum_{k=1}^{K}\left\{m_k(y_k)\frac{\partial f}{\partial y_k} + \frac{v_k(y_k)}{2}\frac{\partial^2 f}{\partial y_k^2}\right\} \quad (18)$$

Then the partial differential equation

$$\frac{\partial f(\mathbf{y},t)}{\partial t} = \mathbf{L}^*\cdot f(\mathbf{y},t) \quad (19)$$

is the so-called *backward* diffusion equation.

3 An Ornstein-Uhlenbeck Process Approximation

The diffusion equations (16) and (19) have a simpler structure than a general multi-dimensional diffusion equation (see e.g. Cox and Miller [1965]) in the sense that there are no cross-coupling terms $\frac{\partial}{\partial y_j \partial y_k}$. This is not unexpected since the K components of the vector process $\mathbf{J}(t) = [J_1(t), J_2(t), \cdots, J_K(t)]$ are independent of each other, hence we find the product form solution of Eq.(8). Therefore, we can proceed to solve the individual diffusion processes $\{Y_k(t)\}$ separately. The forward equation for the k-th component process $Y_k(t)$ is given from Eq.(16) as

$$\frac{\partial f_k(y,t)}{\partial t}$$
$$= -\frac{\partial}{\partial y}[m_k(y)f_k(y,t)] + \frac{1}{2}\frac{\partial^2}{\partial y^2}[v_k(y)f_k(y,t)] \quad (20)$$

Now let y_k^* be such that $m_k(y)$ of Eq.(14), known as the infinitesimal mean, becomes zero, i.e.,

$$y_k^* = \frac{N_k\alpha_k}{\alpha_k + \beta_k} \quad (21)$$

Thus we can write

$$m_k(y) = -(\alpha_k + \beta_k)(y - y_k^*). \quad (22)$$

If we consider a narrow region around $y = y_k^*$ (which will be well justified as discussed later), we can approximate $v_k(y)$, the infinitesimal variance, by a constant value

$$v_k(y) \approx v_k(y_k^*) = \frac{2N_k\alpha_k\beta_k}{\alpha_k + \beta_k}. \quad (23)$$

Then the diffusion equation (20) becomes

$$\frac{\partial f_k(y,t)}{\partial t} = (\alpha_k + \beta_k)\frac{\partial}{\partial y}[(y - y_k^*)f_k(y,t)]$$
$$+ (\frac{N_k\alpha_k\beta_k}{\alpha_k + \beta_k})\frac{\partial^2}{\partial y^2}f_k(y,t). \quad (24)$$

The diffusion process that is characterized by the above equation is called an Ornstein-Uhlenbeck process (Feller [1966], Cox and Miller [1965]). This type of equation appeared in an analysis of the ALOHA random-access protocol (see Kobayashi, Onozato and Huynh [1977], Kobayashi [1983]). The equilibrium-state density function, denoted $f_k(y)$, is shown to be

$$f_k(y) = \lim_{t\to\infty} f_k(y,t) = \frac{1}{\sqrt{2\pi\sigma_k^2}}exp\{-\frac{(y - y_k^*)^2}{2\sigma_k^2}\}, \quad (25)$$

where

$$\sigma_k^2 = \frac{v_k(y_k^*)}{2(\alpha_k + \beta_k)} = \frac{N_k\alpha_k\beta_k}{(\alpha_k + \beta_k)^2}. \quad (26)$$

When we impose the reflecting boundaries at $y = 0$ and $y = N_k$, the corresponding solution is a truncated (at $y = 0$ and $y = N_k$) Gaussian distribution of the form (25).

4 Analysis of the Buffer Behavior

Now let us return to the generalised birth-and-death process equation (7) or its diffusion process analog (12). The buffer content $Q(t)$, a fluid process, is coupled with the vector process $\mathbf{J}(t)$ (or its approximation $\mathbf{Y}(t)$) only through its weighted sum $R(t)$. Because we are interested in the equilibrium state solution, we can write the diffusion equation (12) in the limit $t \to \infty$ as

$$\begin{aligned} &(\textstyle\sum_{k=1}^{K} R_k y_k - C)\frac{\partial f(\mathbf{y},x)}{\partial x} \\ = \;& \textstyle\sum_{k=1}^{K} \Big[(\alpha_k+\beta_k)\frac{\partial}{\partial y_k}[(y_k - y_k^*)f(\mathbf{y},x)] \\ &+(\frac{N_k\alpha_k\beta_k}{\alpha_k+\beta_k})\frac{\partial^2}{\partial y_k^2}f(\mathbf{y},x)\Big], \end{aligned} \tag{27}$$

where

$$f(\mathbf{y},x) \stackrel{\text{def}}{=} \lim_{t\to\infty} f(\mathbf{y},x,t). \tag{28}$$

We may solve the partial differential equation (27) by separating the variables $\mathbf{y}$ and x, and write

$$f(\mathbf{y},x) = g(\mathbf{y}) \cdot F(x). \tag{29}$$

Then Eq.(27) reduces to the following ordinary differential equation.

$$\frac{F'(x)}{F(x)} = \frac{\mathbf{L}\cdot g(\mathbf{y})}{(\sum_{k=1}^{K} R_k y_k - C) g(\mathbf{y})} = u \tag{30}$$

where $\mathbf{L}$ is the linear operator defined by Eq.(17) and u is a real constant to be determined below. From the first and last terms in Eq.(30) we readily find

$$F(x) = e^{u\cdot x} \tag{31}$$

except for a multiplicative constant, which we shall determine later. The differential equation for $g(\mathbf{y})$ is obtained from the second and last terms in Eq.(30), by explicitly writing the operator $\mathbf{L}$:

$$\begin{aligned} &u(\textstyle\sum_{k=1}^{K} R_k y_k - C) g(\mathbf{y}) \\ = \;& \textstyle\sum_{k=1}^{K} \Big\{(\alpha_k+\beta_k)\frac{\partial}{\partial y_k}[(y_k - y_k^*)g(\mathbf{y})] \\ &+(\frac{N_k\alpha_k\beta_k}{\alpha_k+\beta_k})\frac{\partial^2}{\partial y_k^2}g(\mathbf{y})\Big\} \end{aligned} \tag{32}$$

We further assume the following separation of variables

$$g(\mathbf{y}) = \prod_{k=1}^{K} g_k(y_k), \tag{33}$$

and partition the constant C as

$$C = \sum_{k=1}^{K} C_k. \tag{34}$$

Then we can write separate differential equations for the individual y_k's by rearranging Eq.(32)

$$\begin{aligned} &(\frac{N_k\alpha_k\beta_k}{\alpha_k+\beta_k})\frac{d^2}{dy_k^2} g_k(y_k) + (\alpha_k+\beta_k)\frac{d}{dy_k}[(y_k - y_k^*)g_k(y_k)] \\ &\qquad - u(R_k y_k - C_k) g_k(y_k) = 0, \end{aligned} \tag{35}$$

for $k = 1, 2, \cdots, K$.

Let us transform the variable y_k in the following way

$$z_k = \frac{y_k - y_k^*}{\sigma_k}, \tag{36}$$

$$\omega_k = z_k - \frac{2R_k\sigma_k}{\alpha_k+\beta_k}u. \tag{37}$$

and define $A_k(z_k)$ and $B_k(\omega_k)$ as

$$g_k(y_k) = g_k(y_k^* + \sigma_k z_k) \stackrel{\text{def}}{=} A_k(z_k)\cdot e^{-z_k^2/4} \tag{38}$$

$$B_k(\omega_k) \stackrel{\text{def}}{=} A_k(\omega_k + \frac{2R_k\sigma_k}{\alpha_k+\beta_k}u). \tag{39}$$

Then Eq.(35) can be transform into

$$\frac{d^2 B_k(\omega_k)}{d\omega_k^2} + \{\frac{1}{2} + v_k(u) - \frac{\omega_k^2}{4}\} B_k(\omega_k) = 0, \tag{40}$$

where

$$v_k(u) = (\frac{R_k\sigma_k}{\alpha_k+\beta_k})^2 u^2 + (\frac{C_k - R_k y_k^*}{\alpha_k+\beta_k})u. \tag{41}$$

Equation (40) is the differential equation for a special parabolic cylinder function, which is also known as Weber's equation (see e.g., Olver [1974] p.206). Its elementary solutions which have the appropriate behavior as $\omega_k \to \pm\infty$ require $v_k(u)$ to be nonnegative integers, i.e., (see e.g. Sweet and Hardin [1970])

$$v_k(u) = i_k, \qquad i_k = 0, 1, 2, \cdots \tag{42}$$

Then the solution of Eq.(40), corresponding to i_k, is given by

$$B_k(\omega_k) = D_{i_k}(w_k) \stackrel{\text{def}}{=} 2^{-\frac{i_k}{2}} e^{-\frac{w_k^2}{4}} H_{i_k}(\frac{w_k}{\sqrt{2}}), \tag{43}$$

where $H_{i_k}(\cdot)$ is the Hermite polynomial of order i_k.

Now we shall determine the eigenvalue u. From Eqs.(34), (41) and (42), it is not difficult to see that u must satisfy the following quadratic equation for each integer vector $\mathbf{i} = (i_1, i_2, \cdots, i_K)$:

$$(\sum_{k=1}^{K} \frac{R_k^2\sigma_k^2}{\alpha_k+\beta_k})u^2 + (C - \sum_{k=1}^{K} R_k y_k^*)u - \sum_{k=1}^{K} i_k(\alpha_k+\beta_k) = 0. \tag{44}$$

Since all the components i_k are nonnegative, Eq.(44) yields two real roots; one positive denoted $u_{\mathbf{i}}^+$, and one negative denoted $u_{\mathbf{i}}^-$. In particular, when $\mathbf{i} = \mathbf{0}$, we find

$$u_0^+ = 0, \tag{45}$$

$$u_0^- = -\frac{C - \sum_{k=1}^{K} R_k y_k^*}{\sum_{k=1}^{K} \frac{R_k^2\sigma_k^2}{\alpha_k+\beta_k}} < 0. \tag{46}$$

The last inequality is obtained by assuming that the system is stable.

Let us assume for the moment that the capacity of the statistical multiplexer output buffer is infinite. Then Eq.(31) suggests that an inclusion of any positive root $u_i^+ > 0$ would lead to an unstable solution. Thus, the general solution of Eq.(29) is representable as

$$f(\mathrm{y}, x) = F(x) \prod_{k=1}^{K} g_k(y_k) = \prod_{k=1}^{K} \frac{e^{-\frac{(y_k - y_k^*)^2}{2\sigma_k^2}}}{\sqrt{2\pi\sigma_k^2}}$$

$$+ \sum_{i} a_i e^{u_i^- x} \prod_{k=1}^{K} e^{-\frac{(y_k - y_k^*)^2}{4\sigma_k^2}} D_{i_k}\left(\frac{y_k - y_k^*}{\sigma_k} - \frac{2R_k\sigma_k}{\alpha_k + \beta_k} u_i^-\right) \quad (47)$$

Note that the first term is the marginal distribution in the limit $x \to +\infty$. A dominant term in the summed terms is the one that corresponds to the root u_0^- of (46), since this root is the largest (i.e., the closest to the origin $u = 0$) among all the negative roots u_i^-.

Then the equilibrium probability distribution function that the buffer content exceeds x is defined by

$$\lim_{t \to +\infty} F(Q(t) > x) = 1 - \underbrace{\int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty}}_{K} f(\mathrm{y}, x)\, dy$$

$$= \sum_{i} b_i e^{u_i^- x} \quad (48)$$

where

$$b_i = -a_i \left(\prod_{k=1}^{K} \sqrt{2\pi\sigma_k^2} \left(-\frac{R_k\sigma_k}{\alpha_k + \alpha_k} u_i^-\right)^{i_k} e^{-\frac{\left(\frac{R_k\sigma_k}{\alpha_k+\beta_k} u_i^-\right)^2}{2}} \right). \quad (49)$$

The above expression for b_i is derived using the property (Gradshteyn and Ryzhik [1965] p.837) that

$$\int_{-\infty}^{+\infty} e^{-(\eta-\zeta)^2} H_m(\eta) d\eta = \sqrt{\pi}(2\zeta)^m. \quad (50)$$

Determination of the unknown coefficients $\{a_i\}$ or $\{b_i\}$ is found not so simple. The cases discussed by Knessl and Morrison [1991] and Hagan, Doering and Levermore [1989] correspond, in our formulation, to the single type of traffic, i.e., $K = 1$. Their approaches, unfortunately, are not extendable to the multiple type case, in which we are primarily interested. We are currently investigating an alternative method to determine the coefficients $\{a_i\}$ (or $\{b_i\}$).

However, the asymptotic behavior of buffer overflow, i.e., for sufficiently large x, is characterized by the term that has the largest negative (i.e. the dominant) root in its exponential term. In our case, this dominant root is u_0^-. Thus we find

$$F(Q > x) \approx b_0 e^{u_0^- x}, \text{ when } x \text{ is large.} \quad (51)$$

Note that $u_0^- \to 0^-$ when traffic intensity $\rho\ (= \frac{\sum_{k=1}^{K} R_k y_k^*}{C})$ approaches to 1, and all the other negative roots are upper-bounded. Also, when the traffic becomes heavy (i.e. $\rho \to 1$), the output buffer queue will blow up. This implies $F(Q > x) \approx 1$, for all $x > 0$. Then we find that $b_0 \approx 1$ (but always less than 1),

$$F(Q > x) \approx e^{u_0^- x}, \text{ when } \rho \approx 1 \text{ and } x \text{ is large.} \quad (52)$$

Our numerical study shows that the diffusion approximation method presented above tends to underestimate the buffer overflow probability. An intuitive interpretation of this error is that the arrival process

$$R_t = \sum_{k=1}^{K} R_k Y_k(t), \quad (53)$$

which is a diffusion approximation of $R(t)$ of Eq.(4), may take negative values, whereas $R(t)$ does not. But such error should become negligible as the traffic intensity increases. In fact we can show that for the infinite source model with single type of traffic (i.e. $K = 1, N \to \infty, \alpha \to 0$ and $N\alpha \to \lambda$), $z_{dom} = \rho u_0^-$, where z_{dom} is the largest negative eigenvalue derived by Kosten [1974]. The arrival process R_t has, in the equilibrium, a Gaussian distribution with mean $m_R (= \sum_{k=1}^{K} R_k y_k^*)$ and variance $\sigma_R^2 (= \sum_{k=1}^{K} R_k^2 \sigma_k^2)$. By applying the theory of large deviations [2], we can show

$$Prob\{R > C\} \approx \frac{1}{\theta^* \sqrt{2\pi\sigma_R^2}} e^{-\frac{\sigma_R^2}{2}\theta^{*2}} \quad (54)$$

where $\theta^* = \frac{C - m_R}{\sigma_R^2}$. This leads to an approximate lower bound for $F(Q > 0)$ since $\{R_t > C\}$ implies $\{Q(t) > 0\}$. By combining the above arguments, we find

$$\left(\frac{1}{\theta^* \sqrt{2\pi\sigma_R^2}} e^{-\frac{\sigma_R^2}{2}\theta^{*2}} \right) e^{u_0^- x} \quad (55)$$

provides a lower bound for $F(Q > x)$ for sufficiently large x.

5 Numerical Examples and Discussions

Now we proceed to apply the above solution method to some numerical examples. As we noted earlier, the heavy-traffic analysis by Knessl and Morrison [1991] corresponds to the diffusion approximation model for a single type of traffic, i.e., $K = 1$. They discuss a number of numerical examples, and compare their asymptotic analysis with the exact solution by Anick, Mitra and Sondhi [1982], and show that the solutions agree quite well.

Such an excellent agreement suggests that our diffusion approximation process model for multiple type of traffic will be also accurate enough for practical use over a wide range of traffic levels although a rigorous proof and numerical validation are yet to be demonstrated.

We choose the following model parameters to be comparable with those of Kosten's numerical analysis, i.e., $K = 2, C = 38$,

Type 1 Sources : $N_1 = 25,\ \alpha_1 = 0.4,\ \beta_1 = 1.5\ ,\ R_1 = 2\ .$

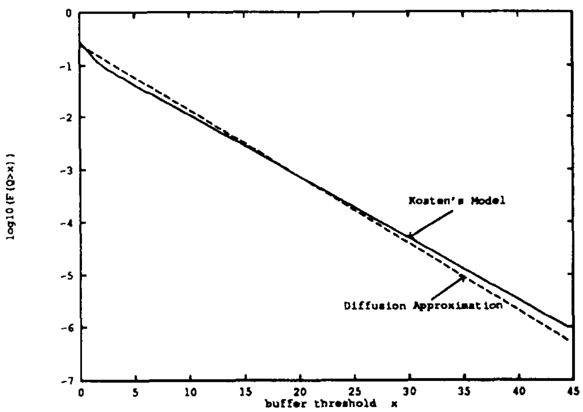

Figure 1: Asymptotic buffer overflow probabilities: Kosten's simulation result vs. diffusion model with two types of traffic.

$$\text{Type 2 Sources : } N_2 = 50,\ \alpha_2 = 0.6,\ \beta_2 = 0.75,\ R_2 = 1\ .$$

For this set of parameters, which results in the traffic intensity $\rho = 0.862$, Kosten's largest negative (i.e. the dominant) eigenvalue is $z_{dom} = -0.270$, while our dominant eigenvalue (Eq.(46)) is $u_0^- = -0.293$. The ratio of z_{dom} to u_0^- is 0.922 and approximately equal to the traffic intensity 0.862. Equation (54) provides $Prob\{R > 38\} \approx 0.254$, which is a good approximation to Kosten's $F(Q > 0)$ obtained by simulation. In Figure 1, the asymptotic buffer overflow probabilities are compared under Kosten [1984] model and our diffusion approximation by Eq.(55).

A more comprehensive numerical study, which will involve extensive simulation efforts, is currently pursued.

The diffusion approximation method that we have introduced can apply to more general models. For example, the exponential distribution assumption made in Section 2 is not necessary (see Kobayashi [1990]). The only restriction is that the Laplace transform of the distribution function of $J(t)$ must be a rational function. That is, if the mean values of the silence and burst periods are given by α_k^{-1} and β_k^{-1}, respectively, the equilibrium distribution of $P(\mathbf{j}, t)$ is given by Eq.(8) irrespective of the distribution forms of silence and burst periods.

6 Conclusion

In this paper, we have developed a multi-dimensional diffusion model to characterize an ATM statistical multiplexer with multiple types of traffic. Our results show the relation between the well-studied statistical multiplexer model and its diffusion approximation. The solution form for the buffer distribution obtained by the diffusion approximation has a spectral expansion expression, whose exponential terms can be easily derived. The diffusion model provides an accurate approximation in the operating region of our interest, i.e., when the number of sources is large and traffic intensity is high.

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