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# A Mathematical Theory for Transient Analysis of Communication Networks*

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**SUMMARY** In the present paper we present a mathematical theory for the transient analysis of probabilistic models relevant to communication networks. First we review the $z$-transform method, the matrix method, and the Laplace transform, as applied to a class of birth-and-death process model that is relevant to characterize network traffic sources. We then show how to develop *transient* solutions in terms of the eigenvalues and spectral expansions. In the latter half the paper we develop a general theory to solve dynamic behavior of statistical multiplexer for *multiple types* of traffic sources, which will arise in the B-ISDN environment. We transform the partial differential equation that governs the system into a concise form by using the theory of *linear operator*. We present a closed form expression (in the Laplace transform domain) for transient solutions of the joint probability distribution of the number of *on* sources and *buffer content* for an arbitrary initial condition. Both finite and infinite buffer capacity cases are solved exactly. The essence of this general result is based on the unique determination of unknown boundary conditions of the probability distributions. Other possible applications of this general theory are discussed, and several problems for future investigations are identified.

***key words:*** *tyansient analysis, statistical multiplexer, multiple-types of traffic, fluid-approximation Kronecker sun, Laplace transform*

## 1. Introduction

A major performance issue associated with future high-speed communication networks is that the transmission speed is so high that a ratio of the propagation delay (determined by the speed of light or electromagnetic wave for a given transmission media) to the packet or cell transmission may become significantly greater than unity. This presents a host of new challenges, as well as opportunities, to network designers and performance analysts. One such issue is that the network design and control methods based on the analysis of *steady state solutions* will no longer be adequate: we need instead to develop a new methodology to formulate and understand the *transient behavior* of network systems.

For a multiplexer in fast packet switching, its workload is characterized by the aggregate packet arrival process that result from the superposition of packet streams from multiple sources. The instantaneous arrival rate in the aggregate packet arrival process is a function of the number of sources in their burst states, and thus fluctuates with high variability. Therefore, the aggregate traffic is usually far from a renewal process, because there is a positive dependence between successive arrival times. This dependence is a major cause of congestion in the multiplexer queue and often leads to surprisingly large packet delays under heavy traffic conditions.

In the present paper we present a mathematical theory for the transient analysis of probabilistic models relevant to communication networks. An emphasis will be placed on a general theory to understand dynamic behavior of multiple traffic source models, statistical multiplexer for B-ISDN. We use a linear operator theory and its spectral expansion method, as applied to the transform domain (the joint $z$-transform and double Laplace transforms) of the partial differential equation that governs the probabilistic behavior of such systems.

## 2. Transient Analysis of On-Off Sources

### 2.1 The $z$-Transform Method

Let us assume that a multiplexer is connected to a number of sources and that $N$ sources are in the "off-hook" state, i.e., $N$ sources are engaged in their calls. Of course, $N$ will change whenever a call is completed or a new call is initiated, but its time constant is on the order of minutes as compared with hundreds of milliseconds for bursts and silence periods, or tens of milliseconds or less for packets. Thus, $N$ can be treated as a fixed constant in the analysis of our model at either burst level or packet level.

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Let $J(t)$ represent the total number of sources in the burst state. If we assume, for mathematical simplicity, the exponential distributions for both burst and silence periods, the random process $J(t)$ can be formulated as a *birth-and-death process*.

Let

$$P(j, t) = P[J(t) = j], \quad 0 \leqq j \leqq N, \tag{1}$$

and let $\lambda(j)$ and $\mu(j)$ be the birth and death rates, respectively, when $J(t) = j$. Then we find that $P(j, t)$ satisfies the following set of differential equations:

$$\frac{dP(j,t)}{dt} = -\{\lambda(j) + \mu(j)\} P(j, t) + \lambda(j-1) P(j-1, t) + \mu(j+1) P(j+1, t), \quad \text{for } 0 \leqq j \leqq N, \tag{2}$$

with

$$P(-1, t) = P(N+1, t) = 0, \quad \text{for all } t. \tag{3}$$

By setting the left side of (2) equal to zero, the equilibrium state solution of Eq. (2) is obtained in a straightforward manner (see e.g., Ref. (13)).

The time-dependent solution for the general class of birth-and-death process model is difficult. But in the on-off source model discussed above, the $\lambda(j)$ and $\mu(j)$ are given by

$$\lambda(j) = (N-j)\alpha, \tag{4}$$

$$\mu(j) = j\beta. \tag{5}$$

where

$$\alpha^{-1} = \text{Mean } \textit{off} \text{ period} \tag{6}$$

$$\beta^{-1} = \text{Mean } \textit{on} \text{ period} \tag{7}$$

and we can obtain a closed form solution of the time-dependent solution as outlined below.

If we define the following $z$-transform with respect to the integer variable $j$

$$G(z, t) = \mathscr{Z}\, P(j, t) = \sum_{j=0}^{N} P(j, t) z^j, \tag{8}$$

which is also known as the *probability generating function*, $E[z^{J(t)}]$. Then the set of $(N+1)$ differential equations of (2) reduces to the following *single* partial differential equation:

$$\frac{\partial G(z,t)}{\partial t} = (z-1)\left[N\alpha G(z, t) - (\alpha z + \beta) \frac{\partial G(z,t)}{\partial z}\right], \tag{9}$$

which can be viewed as a special case of the following planar differential equation:

$$p \frac{\partial G}{\partial t} + q \frac{\partial G}{\partial z} = r, \tag{10}$$

where $G$ denotes $G(z, t)$ defined above, and $p$, $q$ and $r$ are, in general, functions of $t$, $z$ and $G$.

By solving the above partial differential equation, we obtain

$$G(z, t) = (\alpha z + \beta)^N \cdot \left[\frac{(\alpha+\beta)(\alpha z + \beta)}{(\alpha z + \beta) - \alpha(z-1) e^{-(\alpha+\beta)t}}\right]^{-N} = [1 + (z-1) q(t)]^N, \tag{11}$$

where

$$q(t) = \frac{\alpha}{\alpha+\beta}(1 - e^{-(\alpha+\beta)t}). \tag{12}$$

In deriving Eq. (11), we assumed that all the sources are initially off, i.e., $J(0) = 0$ with probability one:

$$G(z, 0) = 1, \tag{13}$$

By taking the inverse $z$-transform of $G(z, t)$, i.e., by calculating the coefficient of the $z^j$ term, we find

$$P(j, t) = \mathscr{Z}^{-1}\{G(z, t)\} = \binom{N}{j} q(t)^j [1 - q(t)]^{N-j}, \tag{14}$$

which is the binomial distribution with parameter $q(t)$.

If we assume, instead of (13), an arbitrary initial condition $\{P(i, 0), 0 \leqq i \leqq N\}$, we obtain

$$G(z, t) = \sum_{i=0}^{N} P(i, 0) \cdot [1 + (z-1) r(t)]^i [1 + (z-1) q(t)]^{N-i}, \tag{15}$$

where $r(t)$ is defined by

$$r(t) = \frac{\alpha}{\alpha+\beta} + \frac{\beta}{\alpha+\beta} e^{-(\alpha+\beta)t} = 1 - \frac{\beta}{\alpha} q(t). \tag{16}$$

The inversion of the generating function $G(z, t)$ of Eq. (15) can be easily performed by observing that

$$G_i(z, t) = [1 + (z-1) r(t)]^i [1 + (z-1) q(t)]^{N-i} \tag{17}$$

is a product of the generating functions of two Bernoulli (or binomial) distributions. Thus we find

$$P_i(j, t) = P[J(t) = j | J(0) = i] = \mathscr{Z}^{-1}\{G_i(z, t)\} = \sum_{k=0}^{j} \binom{i}{j-k} r(t)^{j-k} [1 - r(t)]^{i-j+k} \cdot \binom{N-i}{k} q(t)^k [1 - q(t)]^{N-i-k}, \tag{18}$$

in which we assume $\binom{i}{j} = 0$ for $i < j$. Therefore we finally obtain

$$P(j, t)=\sum_{i=0}^{N} P(i, 0) P_i(j, t) \tag{19}$$

Now in the limit $t \rightarrow \infty$, we find both $q(t)$ and $r(t)$ converge to $\alpha/(\alpha+\beta)$:

$$\lim_{t \to \infty} r(t)=\lim_{t \to \infty} q(t)=\frac{\alpha}{\alpha+\beta} \tag{20}$$

Thus we find

$$\lim_{t \to \infty} G(z, t)=\left[1+(z-1)\frac{\alpha}{\alpha+\beta}\right]^N, \tag{21}$$

and

$$\lim_{t \to \infty} P(j, t)=\mathscr{Z}^{-1}\left\{\lim_{t \to \infty} G(z, t)\right\}$$
$$=\binom{N}{j}\left(\frac{\alpha}{\alpha+\beta}\right)^j\left(\frac{\beta}{\alpha+\beta}\right)^{N-j} \overset{\text{def}}{=} p_j, \tag{22}$$

which is the equilibrium solution, independent of the initial distribution $\{P(i, 0)\}$.

### 2.2 Matrix Representation, Spectral Analysis and the Laplace Transform

Now we discuss a matrix representation approach to the birth-and-death process model. We draw much of the material from Syski[29]. The matrix approach is applicable to a general class of the birth-and-death process model, whereas the $z$-transform method discussed in the previous section is suitable for a limited class of the models, although it often results in a nice closed form solution as shown in Eqs. (14) and (18). The on-off source model in which the birth and death rates take special forms of Eqs. (4) and (5) happens to be one of few cases that allow such elegant solutions.

Define a column vector of $(N+1)$ dimension:

$$\boldsymbol{P}(t)=\begin{bmatrix} P(0, t) \\ P(1, t) \\ \vdots \\ P(j, t) \\ \vdots \\ P(N, t) \end{bmatrix} \tag{23}$$

Then Eq. (2) can be written in matrix form as

$$\frac{d\boldsymbol{P}(t)}{dt}=\mathscr{M}\,\boldsymbol{P}(t), \tag{24}$$

where $\mathscr{M}$ is an $(N+1)\times(N+1)$ tridiagonal matrix whose elements $m_{ij}$ are defined as

$$m_{jj-1}=\lambda(j-1),\ m_{jj}=-\lambda(j)-\mu(j),$$
$$m_{jj+1}=\mu(j+1)$$
$$\text{for } j=0, 1, \cdots, N,$$
$$\text{and } m_{ij}=0 \text{ for all other } i \text{ and } j. \tag{25}$$

The differential equation (24) has the solution:

$$\boldsymbol{P}(t)=\mathscr{P}(t)\,\boldsymbol{P}(0), \tag{26}$$

where the matrix function $\mathscr{P}(t)$ is given by

$$\mathscr{P}(t)=e^{\mathscr{M}t}. \tag{27}$$

The matrix $\mathscr{M}$, called the *infinitesimal generator* of the Markov process $J(t)$, is a singular matrix, because the sums of its rows equal to zero, therefore its determinant vanishes. The matrix $\mathscr{P}(t)$ is a *stochastic* matrix, and satisfies the following Markov transition property, i.e., the Chapman-Kolmogorov equation:

$$\mathscr{P}(t+h)=\mathscr{P}(t)\mathscr{P}(h), \tag{28}$$

Assuming that the Markov process is *ergodic* and hence that there exist stationary distributions, we denote by $\boldsymbol{p}$ the vector of the limiting distribution:

$$\boldsymbol{p}=\lim_{t \to \infty} \boldsymbol{P}(t), \tag{29}$$

whose $j$th element is $p_j$ as defined in Eq. (22).

It then follows from Eq. (24) that for the stationary distribution

$$\mathscr{M}\,\boldsymbol{p}=0. \tag{30}$$

Let $\{s_j, j=0, 1, \cdots, N\}$ be eigenvalues of $\mathscr{M}$, i.e., they satisfy the determinant equation:

$$\det|\mathscr{M}-sI|=0, \tag{31}$$

then eigenvalues of $\mathscr{P}(t)$ are $e^{s_j t}$. Therefore, the determinant of $\mathscr{P}(t)$ is given by

$$\det|\mathscr{P}(t)|=e^{T_r\mathscr{M}t} \tag{32}$$

where $T_r\mathscr{M}$, the trace of $\mathscr{M}$, is the sum of diagonal elements of $\mathscr{M}$:

$$T_r\mathscr{M}=-\sum_{j=0}^{N}[\lambda(j)+\mu(j)]. \tag{33}$$

**Spectral Expansion and Projection Matrices**

It is known from the matrix theory that the trace is the sum of all eigenvalues:

$$T_r\mathscr{M}=\sum_{j=0}^{N} s_j, \tag{34}$$

and that the determinant is the product of all eigenvalues:

$$\det|\mathscr{M}|=\prod_{j=0}^{N} s_j. \tag{35}$$

Since $\mathscr{M}$ is singular as we remarked earlier, the determinant of $\mathscr{M}$ is zero. Therefore at least one eigenvalue of $\mathscr{M}$ is zero, and thus the corresponding eigenvalue of $\mathscr{P}(t)$ is unity. Suppose that all eigenvalues of $\mathscr{P}(t)$ are distinct. Then the the matrix has the following expansion, which is referred to as the *spectral expansion*:

$$\mathcal{P}(t)=\sum_{j=0}^{N} e^{s_j t}\mathcal{E}_j, \tag{36}$$

where $\mathcal{E}_j$ is a *projection* matrix with the properties:

$$\mathcal{E}_j\mathcal{E}_k=\begin{cases}0 & \text{for } k\neq j\\ \mathcal{E}_j & \text{for } k=j,\end{cases} \tag{37}$$

and

$$\sum_{j=0}^{N}\mathcal{E}_j=I. \tag{38}$$

When some of eigenvalues are equal, i.e., when the characteristic equation (31) has multiple roots, the corresponding spectral expansion of Eq. (36) includes polynomials in $t$.

Let $U_j$ and $V_j$ be left and right eigenvectors of $\mathcal{M}$ associated with the eigenvalue $s_j$ such that

$$U_j'\mathcal{M}=s_jU_j', \tag{39}$$

$$\mathcal{M}\,V_j=s_jV_j, \tag{40}$$

$$U_j'V_k=\delta_{jk}=\begin{cases}1, & \text{for } k=j,\\ 0, & \text{for } k\neq j.\end{cases} \tag{41}$$

Then we can write

$$\mathcal{E}_j=V_jU_j'. \tag{42}$$

By substituting Eq. (36) into Eq. (26), the probability distribution vector $P(t)$ can be written as

$$P(t)=\sum_{j=0}^{N} a_j e^{s_j t}V_j, \tag{43}$$

where the $a_j$ is the inner product of the initial distribution vector $P(0)$ and the right eigenvector:

$$a_j=U_j'P(0). \tag{44}$$

It can be shown that the real eigenvalues are zero or negative, whereas the complex eigenvalues have always negative real parts. The number of zero eigenvalue is $N+1-r$, where $r$ is the rank of the matrix $\mathcal{M}$. When the process is ergodic, there should be only one zero eigenvalue, thus $\mathcal{M}$ is of rank $N$. It is not difficult to show that the left eigenvector $U_0$ associated with the eigenvalue $s_0=0$ has unity as its all elements:

$$U_0'=[1, 1, \cdots, 1], \tag{45}$$

which implies $a_0=1$. We can therefore write

$$P(t)=V_0+\sum_{j=1}^{N} e^{s_j t}a_jV_j. \tag{46}$$

Hence the limiting value of $P(t)$, as $t$ tends to infinity, is $V_0$, because all the exponential terms approach zero. As is expected from Eq. (29), we find

$$p=V_0 \tag{47}$$

It will be instructive to note that the limiting distribution $p$ can also be expressed as

$$p=\mathcal{E}_0P(0), \tag{48}$$

which is the projection acting on the initial distribution.

**Application to the On-Off Source Model**

Now we are ready to apply the above results to the on-off source model, where the birth and death rates are given by Eqs. (4) and (5) and the matrix $\mathcal{M}$ of Eq. (25) is given by

$$\mathcal{M}=\begin{bmatrix} -N\alpha & \beta & & & & \\ N\alpha & -(N-1)\alpha-\beta & 2\beta & & & \\ & (N-1)\alpha & -(N-2)\alpha-2\beta & & & \\ & \ddots & \vdots & \ddots & & \\ & & & & -\alpha-(N-1)\beta & N\beta \\ & & & & \alpha & -N\beta \end{bmatrix}$$

We then have

$$\det|\mathcal{M}-sI|=\prod_{j=0}^{N}[j(\alpha+\beta)+s] \tag{49}$$

so the eigenvalues of $\mathcal{M}$ are

$$s_j=-j(\alpha+\beta),\quad j=0, 1, 2, \cdots, N \tag{50}$$

and the non-equilibrium distribution $P(t)$ can be written as follows:

$$P(t)=p+\sum_{j=1}^{N} e^{-j(\alpha+\beta)t}a_jV_j \tag{51}$$

**An Infinite Source Model**

Consider the limiting case, where

$$N\to\infty,\ \alpha\to 0,\quad \text{while}\quad N\alpha\to\lambda, \tag{52}$$

Then the source model becomes an infinite source model (see e.g. Ref. (13)), and bursts arrive according to a Poisson process with rate $\lambda$ and lasts on the average for $1/\beta$ seconds. The transient probability distribution function now takes the form

$$P(j, t)=\frac{\left\{\frac{\lambda}{\beta}(1-e^{-\beta t})\right\}^j}{j!}\exp\left\{-\frac{\lambda}{\beta}(1-e^{-\beta t})\right\},\quad j=0, 1, \cdots \tag{53}$$

The last expression represents the probability that $j$

bursts are found at time $t$, and it is equivalent to the time-dependent solution for an $M/M/\infty$ queueing system. Using a result in Takaĉs[30], we can assume that the burst period has a *general distribution* $G(t)$, and find the following transient solution:

$$P(j,t)=\frac{\left\{\lambda\int_0^t G^c(y)\,dy\right\}^j}{j!}\exp\left\{-\lambda\int_0^t G^c(y)\,dy\right\}, \tag{54}$$

where $G^c(t)$ is the complement of the distribution function $G(t)$.

The infinitesimal generator matrix $\mathcal{M}$ is an infinite matrix given by

$$\mathcal{M}=\begin{bmatrix} -\lambda & \beta & 0 & 0 & \cdots \\ \lambda & -(\lambda+\beta) & 2\beta & 0 & \cdots \\ 0 & \lambda & -(\lambda+2\beta) & 0 & \cdots \\ 0 & 0 & \lambda & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}. \tag{55}$$

The above result carries over to the multiple types of traffic, as discussed by Kobayashi and Ren[18].

**The Laplace Transform Method**

Let us begin with the differential Eq. (24), i.e.

$$\frac{d\boldsymbol{P}(t)}{dt}=\mathcal{M}\,\boldsymbol{P}(t). \tag{56}$$

We define the Laplace transform of $\boldsymbol{P}(t)$ as

$$\boldsymbol{P}^*(s)=\mathcal{L}_t\{\boldsymbol{P}(t)\}=\int_0^\infty \boldsymbol{P}(t)\,e^{-st}dt. \tag{57}$$

Then Eq. (24) is transformed into the following matrix equation:

$$s\boldsymbol{P}^*(s)-\boldsymbol{P}(0)=\mathcal{M}\,\boldsymbol{P}^*(s), \tag{58}$$

which leads to

$$\boldsymbol{P}^*(s)=\mathcal{R}(s)\,\boldsymbol{P}(0), \tag{59}$$

where

$$\mathcal{R}(s)=[sI-\mathcal{M}]^{-1}=\mathcal{L}_t\{\mathcal{P}(t)\}. \tag{60}$$

This relation between $\mathcal{R}(s)$ and $\mathcal{P}(t)$ is readily obtainable from Eq. (26). The Laplace transform $\mathcal{R}(s)$ is calld the resolvent of the Markov transition operator $\mathcal{P}(t)$ (see Feller[7], Syski[29]). The inverse matrix Eq. (60) exists for those $s$ which are not equal to the eigenvalues $s$'s of $\mathcal{M}$. The set of all $s$ for which $\mathcal{R}(s)$ exists is called the *resolvent set* and the set of eigenvalues is called the *spectrum*. Referring to the spectral expansion of Eq. (36), we have the following expansion:

$$\mathcal{R}(s)=\sum_{j=0}^{N}\frac{\mathcal{E}_j}{s-s_j}, \tag{61}$$

which leads to the following relationship between the resolvent and the projection matrix:

$$\mathcal{E}_j=\lim_{s\to s_j}\mathcal{R}(s)(s-s_j). \tag{62}$$

We now go back to Eq. (60) and write

$$\mathcal{R}(s)=\frac{\mathcal{A}(s)}{\det|sI-\mathcal{M}|}=\frac{\mathcal{A}(s)}{C(s)}, \tag{63}$$

where

$$\det|sI-\mathcal{M}|=\prod_{j=0}^{N}[j(\alpha+\beta)+s], \tag{64}$$

as seen from Eq. (49). We write the $(N+1)\times(N+1)$ matrix $\mathcal{A}(s)$ defined in Eq. (63) as

$$\mathcal{A}(s)=[A_{ij}(s)], \tag{65}$$

where

$$A_{ij}(s)=\det|\Delta_{ji}|. \tag{66}$$

The matrix $\Delta_{ji}$ is a minor matrix of order $N$, obtained by eliminating the $j$th row and $i$th column of $[sI-\mathcal{M}]$.

## 3. Transient Analysis of a Statistical Multiplexer

There have been a number of studies that report analytic model of statistical multiplexing, but most studies have been limited to models with one type of information source (e.g., voice sources) or two types of sources (e.g, voice and data).

Anick, Mitra and Sondhi[2], Cohen[5], Hashida and Fujiki [9], Kosten [19], Mitra [21], Stern [27] and others[4],[11],[23],[25] discuss *fluid approximation* models that are relevant to statistical multiplexing systems and obtain the *equilibrium state* solutions for *single type* of traffic sources. For an expository treatment of these earlier results, the reader is referred to Kobayashi[15]. Ren and Kobayashi[26] recently obtained *transient solutions* for such statistical multiplexors by using the double Laplace transform method.

Kosten[20] presents some analytic and simulation methods to derive the equilibrium solution for *multiple types* of traffic. Kobayashi[16],[17] discusses the case of *infinite sources* with multiple types, and characterizes an asymptotic behavior of the buffer contents in terms of simple parameters of what he terms the "dominant" type traffic. Elwalid, Mitra and Stern[28] and Stern and Elwalid[28] discuss equilibrium state solutions when the sources are modeled as Markov modulated sources, and derive both theoretical results and computational methods.

In this section we present a general theory of *transient analysis* of a statistical multiplexer with *multiple types* of traffic, based on the recent result by Kobayashi and Ren[18].

### 3.1 Statistical Multiplexer for Multiple Types of Traffic

Let there be $N_m$ sources of type $m$, where $m=1, 2, \cdots, M$, and let $J_m(t)$ denote the number of types $m$ sources in "on" state (or in "burst", or "talk spurt" mode in the case of voice sources). Therefore, there are $N_m - J_m(t)$ sources which are in "off" state (i.e., "silence" mode).

We assume that successive "on" and "off" periods of each source form an alternating renewal process. For mathematical simplicity, we further assume that the "on" and "off" periods of type $m$ sources are both exponentially distributed with parameters $\alpha_m$ and $\beta_m$, respectively:

$$\alpha_m^{-1} = \text{The mean } \textit{off} \text{ period of a type } m \text{ source;} \tag{67}$$

$$\beta_m^{-1} = \text{The mean } \textit{on} \text{ period of a type } m \text{ source.} \tag{68}$$

Let $\boldsymbol{j}$ be a vector defined by

$$\boldsymbol{j} = [j_1, j_2, \cdots, j_M], \tag{69}$$

where $j_m$ is an integer that $J_m(t)$ can take on.

Each source of type $m$ in its *burst* state generates packets (or *cells* in the ATM terminology) at the rate of $R_m$ [packets/unit time]. The aggregate rate of packet arrivals at time $t$ is therefore

$$R(t) = \sum_{m=1}^{M} R_m J_m(t). \tag{70}$$

Suppose that the buffer content is initially empty. Then while $R(t) < C$, the link capacity of the multiplexer output, the arriving packets are processed immediately, thus no queue of packets will develop in the buffer. Once $R(t)$ exceeds $C$, however, the output link can no longer handle all the packets instantaneously, and buffer contents will grow or deplete at the rate $R(t) - C$, depending on whether this quantity is positive or negative at a given instant.

Let us define $Q(t)$ as the total amount of packets found in the buffer of multiplexer output link. Strictly speaking, $Q(t)$ is an integer-valued function, but we approximate it by time-continuous function, assuming that a series of packets arrive like fluid flows. This assumption is well justified in modeling a multiplexer for a high speed link.

By extending Eq. (1) we define the following probability distribution function:

$$P(\boldsymbol{j};\ t, x) = P[J_m(t) = j_m,\ 1 \leqq m \leqq M;\ \text{and } Q(t) \leqq x]. \tag{71}$$

Then we obtain the following partial differential equation that governs the dynamic behavior of the system under discussion:

$$\frac{\partial P(\boldsymbol{j};\ t, x)}{\partial t} + \left(\sum_{m=1}^{M} R_m j_m - C\right)\frac{\partial}{\partial x} P(\boldsymbol{j};\ t, x)$$
$$= -\sum_{m=1}^{M} [(N_m - j_m)\alpha_m + j_m \beta_m] P(\boldsymbol{j};\ t, x)$$
$$+ \sum_{m=1}^{M} (N_m - j_m + 1)\alpha_m P(\boldsymbol{j} - \boldsymbol{1}_m;\ t, x)$$
$$+ \sum_{m=1}^{M} (j_m + 1)\beta_m P(\boldsymbol{j} + \boldsymbol{1}_m;\ t, x), \tag{72}$$

where $\boldsymbol{1}_m$ is a vector that has unity in its $m$th entry and is zero elsewhere.

### 3.2 Linear Operator and Transform Methods

In order to solve Eq. (72) we define the multidimensional $z$-transform or probability generating function

$$G(\boldsymbol{z};\ t, x) = \mathcal{Z}\{P(\boldsymbol{j};\ t, x)\}$$
$$= \sum_{j_1=0}^{N_1}\sum_{j_2=0}^{N_2}\cdots\sum_{j_M=0}^{N_M} P(\boldsymbol{j};\ t, x) z_1^{j_1} z_2^{j_2} \cdots z_M^{j_M}. \tag{73}$$

Substitution of Eq. (72) into Eq. (73) results in the following expression, which is similar to to Eq. (9) (see Kobayashi[16]):

$$\frac{\partial G(\boldsymbol{z};\ t, x)}{\partial t} + \frac{\partial}{\partial x}\left(\sum_{m=1}^{M} R_m z_m \frac{\partial G(\boldsymbol{z};\ t, x)}{\partial z_m} - C G(\boldsymbol{z};\ t, x)\right)$$
$$= \sum_{m=1}^{M} (z_m - 1)\left[N\alpha_m G(\boldsymbol{z};\ t, x) - (\alpha_m z_m + \beta_m)\frac{\partial G(\boldsymbol{z};\ t, x)}{\partial z_m}\right], \tag{74}$$

Equation (74) is rather complex in its appearance, but if we apply the theory of *linear operator*, the above equation can be transformed into the following concise expression:

$$\frac{\partial}{\partial t} G + \mathcal{D}\frac{\partial}{\partial x} G = \mathcal{M}\, G, \tag{75}$$

where we now drop the arguments $\boldsymbol{z}$, $t$ and $x$ in the function $G(\boldsymbol{z};\ t, x)$. $\mathcal{M}$ is the linear operator defined by

$$\mathcal{M}\, G = \sum_{m=1}^{M} (z_m - 1)\left[N_m \alpha_m G - (\alpha_m z_m + \beta_m)\frac{\partial G}{\partial z_m}\right], \tag{76}$$

and the operator $\mathcal{D}$ is defined as

$$\mathcal{D}\, G = \sum_{m=1}^{M}\left[R_m z_m \frac{\partial}{\partial z_m} G - CG\right]. \tag{77}$$

A critical observation that we make here is that the operator $\mathcal{M}$ can be interpreted as a matrix multiplying $G$, which should be interpreted as a vector of dimension $L$, where $L=\prod_{m=1}^{M}(N_m+1)$. The matrix $\mathcal{M}$ is representable in terms of the infinitesimal generator matrices $\mathcal{M}$'s:

$$\mathcal{M}=\mathcal{M}_1\oplus\mathcal{M}_2\oplus\cdots\oplus\mathcal{M}_M, \tag{78}$$

where $\oplus$ represents Kronecker sum. The Kronecker sum of two matrices $\mathcal{A}$ and $\mathcal{B}$ is defined by (see e.g., Bellman[3], Neuts[24])

$$\mathcal{A}\oplus\mathcal{B}=\mathcal{A}\otimes I+I\otimes\mathcal{B} \tag{79}$$

where $\otimes$ is the Kronecker product, which is well known in the matrix theory. Lexicographical ordering of the elements of vector $G$ and those of $\mathcal{M}$ should be chosen consistently and the example will clarify this point. The matrix $\mathcal{M}_m$ is an $(N_m+1)\times(N_m+1)$ tridiagonal matrix of the form $\mathcal{M}$ defined earlier.

Similarly the matrix interpretation of the operator $\mathcal{D}$ is given by the following diagonal matrix of dimension $L$:

$$\mathcal{D}=\mathcal{R}^{(1)}\oplus\mathcal{R}^{(2)}\oplus\cdots\oplus\mathcal{R}^{(M)}-C\cdot I, \tag{80}$$

where $I$ is the identity matrix of dimension $L$, and

$$\mathcal{R}^{(m)}=\mathrm{diag}[0, R_m, \cdots, j_mR_m, \cdots, N_mR_n]. \tag{81}$$

We define the Laplace transform of $G(\mathbf{z};\ t, x)$ with respect to time $t$ by

$$G^*(\mathbf{z};\ s, x)=\mathcal{L}_t\{G(\mathbf{z};\ t, x)\}. \tag{82}$$

Then Eq. (75) can be transformed into

$$sG^*(\mathbf{z};\ s, x)-G(\mathbf{z};\ 0, x)+\mathcal{D}\frac{\partial}{\partial x}G^*(\mathbf{z};\ s, x)$$
$$=\mathcal{M}\,G^*(\mathbf{z};\ s, x). \tag{83}$$

We then take the Laplace transform of $G^*(\mathbf{z};\ s, x)$ with respect to the buffer content variable $x$;

$$sG^{**}(\mathbf{z};\ s, u)-G^*(\mathbf{z};\ 0, u)+\mathcal{D}\{uG^{**}(\mathbf{z};\ s, u)$$
$$-G^*(\mathbf{z};\ s, 0)\}=\mathcal{M}\,G^{**}(\mathbf{z};\ s, u), \tag{84}$$

from which we obtain

$$[sI+u\,\mathcal{D}-\mathcal{M}]G^{**}(\mathbf{z};\ s, u)$$
$$=G^*(\mathbf{z};\ 0, u)+\mathcal{D}\,G^*(\mathbf{z};\ s, 0). \tag{85}$$

We formally define an operator $\mathcal{R}(s)$ by

$$\mathcal{R}(s)=[sI+u\,\mathcal{D}-\mathcal{M}]^{-1}, \tag{86}$$

which is a generalization of the *resolvent* defined in the previous section. By applying the spectral expansion method discussed earlier, we can represent the operator $\mathcal{R}(s)$ as follows:

$$\mathcal{R}(s)=\sum_{j=0}^{L-1}\frac{\mathcal{E}_j}{s-s_j}, \tag{87}$$

where $\mathcal{E}_j$ is the projection operator, and is obtained as

$$\mathcal{E}_j=\lim_{s\to s_j}(s-s_j)\,\mathcal{R}(s). \tag{88}$$

As shown by Eq. (3), the operator $\mathcal{E}_j$ can be more easily obtained, once we find the $j$th left and right eigenvectors associated with the eigenvalue $s_j$.

Let $s$ be one of the eigenvalues and let $V(\mathbf{z};\ u)$ be the multidimensional $z$ transform of the associated right eigenvector. Then $V(\mathbf{z};\ u)$ should satisfy the following characteristic equation:

$$sV(\mathbf{z};\ u)=[\mathcal{M}-u\,\mathcal{D}]V(\mathbf{z};\ u) \tag{89}$$

or

$$\sum_{m=1}^{M}[\alpha_m z_m^2+(uR_m+\beta_m-\alpha_m)z_m$$
$$-\beta_m]\frac{\partial}{\partial z_m}\{\ln V(\mathbf{z};\ u)\}$$
$$=-s+uC+\sum_{m=1}^{M}[N_m\alpha_m(z_m-1)]. \tag{90}$$

Let $z_{m1}(u)$ and $z_{m2}(u)$ be two (real) roots of the quadratic equation

$$\alpha_m z_m^2+(uR_m+\beta_m-\alpha_m)z_m-\beta_m=0, \tag{91}$$

which leads to the following *product form* for $V(\mathbf{z};\ u)$:

$$V(\mathbf{z};\ u)=\prod_{m=1}^{M}(z_m-z_{m1}(u))^{k_m}(z_m-z_{m2}(u))^{N_m-k_m}, \tag{92}$$

where $k_m$ is an integer parameter between 0 and $N_m$. Then taking the logarithm of Eq. (92), and subs itut- ing it into Eq. (90), we find the following ex plicit formula for the eigenvalue:

$$s=uC-\sum_{m=1}^{M}\alpha_m[N_m-k_m z_{m2}(u)$$
$$-(N_m-k_m)z_{m1}(u)]. \tag{93}$$

By substituting

$$z_{m1}(u), z_{m2}(u)=\frac{-(uR_m+\beta_m-\alpha_m)\pm\sqrt{D_m(u)}}{2\alpha_m}, \tag{94}$$

$$D_m=(uR_m+\beta_m-\alpha_m)^2+4\alpha_m\beta_m, \tag{95}$$

into the last equation, we obtain

$$s=u\left(C-\frac{1}{2}\sum_{m=1}^{M}R_mN_m\right)$$
$$-\frac{1}{2}\sum_{m=1}^{M}(\alpha_m+\beta_m)N_m$$
$$-\sum_{m=1}^{M}\sqrt{D_m}\left(k_m-\frac{N_m}{2}\right). \tag{96}$$

Therefore, for a given integer vector

$$\boldsymbol{k}=[k_1, k_2, \cdots, k_M] \tag{97}$$

we uniquely determine the corresponding eigenvalue $s_k$ as a function of $u$, and the associated eigenvector $V_k(\boldsymbol{z}; u)$, which takes the form Eq. (92).

Now we write the multi-dimensional $z$-transform of the $\boldsymbol{k}$th eigenvector as

$$V_k(\boldsymbol{z};\ u)=\prod_{m=1}^{M} V_{k_m}(z_m;\ u), \tag{98}$$

where

$$V_{k_m}(z;\ u)=(z-z_{m1}(u))^{k_m}(z-z_{m2}(u))^{N_m-k_m}. \tag{99}$$

Once the right eigenvector, denoted $\boldsymbol{V}_k$, is obtained, the corresponding left eigenvector $\boldsymbol{U}_k$ is given by

$$U_k=(\mathcal{Q}^{-1})^2 \boldsymbol{V}_k, \tag{100}$$

where $\mathcal{Q}$ is a diagonal matrix that transforms the tridiagonal matrix $\mathcal{M}$ into a symmetric matrix. It is not difficult to show that

$$\mathcal{Q}=\mathcal{Q}_1\otimes\mathcal{Q}_2\otimes\cdots\otimes\mathcal{Q}_M, \tag{101}$$

where $\otimes$ represents Kronecker product and the $j$ element of the diagonal matrix $\mathcal{Q}_m$ is given by

$$\mathcal{Q}_{m_j}=\sqrt{\left(\frac{\alpha_m}{\beta_m}\right)^j\binom{N_m}{j}}, \tag{102}$$

which transforms $\mathcal{M}_m$ into a symmetric matrix.

We normalize these eigenvectors so that

$$\boldsymbol{U}'_k \boldsymbol{V}_{k'}=\delta_{kk'}, \tag{103}$$

where $\delta_{kk'}$ is the Kronecker delta. We find that the projection operators $\mathcal{E}_k$ is representable, in its matrix interpretation, as

$$\mathcal{E}_k=\boldsymbol{V}_k\ \boldsymbol{U}'_k. \tag{104}$$

Therefore, from Eqs. (85) and (87), we obtain the following:

$$\boldsymbol{P}^{**}(s, u)=\mathcal{Z}^{-1}\{G^{**}(\boldsymbol{z};\ s, u)\}$$
$$=\sum_k \frac{\mathcal{E}_k}{s-s_k}[\boldsymbol{P}^{*}(0, u)+\mathcal{D}\ \boldsymbol{P}^{*}(s,0)], \tag{105}$$

where $\mathcal{Z}^{-1}$ is the inverse $z$-transform. Thus the $\boldsymbol{P}^{**}(s, u)$ is the probability vector of dimension $L=\prod_{m=1}^{M}(N_m+1)$, and its $\boldsymbol{k}$th element is the coefficient of $z_1^{k_1}z_2^{k_2}\cdots z_M^{k_M}$ term in $G^{**}(\boldsymbol{z};\ s, u)$, and is equivalent to $\mathcal{L}_t\mathcal{L}_x\{P(\boldsymbol{k};\ t, x)\}$.

Taking the inverse Laplace transform of Eq. (105) with respect to the variable $s$, we obtain the following time-dependent solution:

$$\boldsymbol{P}^{*}(t, u)=\mathcal{L}_s^{-1}\{\boldsymbol{P}^{**}(s, u)\}$$
$$=\sum_k \mathcal{E}_k\Big[\exp\{s_k\ t\}\frac{e^{-ux_0}}{u}\ \boldsymbol{P}(0, +\infty)$$
$$+\mathcal{D}\mathcal{L}_s^{-1}\Big\{\frac{\boldsymbol{P}^{*}(s,0)}{s-s_k}\Big\}\Big], \tag{106}$$

where $x_0$ is the initial buffer content, and the marginal (in the sense that it is independent of the variable $x$) distribution, $\boldsymbol{P}(0, +\infty)$, represents the initial distribution of the "on" sources. In other words $\boldsymbol{P}(0, +\infty)$ is the distribution of the vector Eq. (69) at $t=0$, and is explicitly given, whereas the boundary condition $\boldsymbol{P}^{*}(s,0)$ is unknown. The above summation should be taken over only those values of $\boldsymbol{k}$ for which $s_k\leqq 0$, since positive eigenvalues would yield unstable solutions. Then taking the inverse Laplace transform of Eq. (106) we finally obtain

$$\boldsymbol{P}(t, x)=\mathcal{L}_u^{-1}\{\boldsymbol{P}^{*}(t, u)\} \tag{107}$$

In many practical problems we need to resort to numerical methods to perform the inverse Laplace transform. (see, for example, Kobayashi[13], pp. 73-74 and references therein.) Abate and Whitt[1] give an extensive discussion on the Fourier-series method for inverting transforms of probability distributions.

### 3.3 The Boundary Conditions

We now discuss how the unknown boundary condition $\boldsymbol{P}^{*}(s,0)$ in Eq. (105) should be determined. First, we need to solve the characteristic equation $\det(u\ \mathcal{D}+sI-\mathcal{M})=0$ with respect to $u$. This can be achieved by solving Eq. (96) with respect to $u$. Unfortunately, however, $u$ cannot generally be given in an explicit form, because $D_m$ in Eq. (96) includes $u$. Hence $u$ must be solved numerically. For the integer vector $\boldsymbol{k}$ of Eq. (97) we denote the corresponding eigenvalue by $u_k$.

We denote $\tilde{\boldsymbol{V}}_k$, $\tilde{\boldsymbol{U}}_k$ and $\tilde{\mathcal{E}}_k$ to represent the right, left eigenvectors and projection operators, respectively. Note that they should be distinguished from $\boldsymbol{V}_k$, $\boldsymbol{U}_k$ and $\mathcal{E}_k$ defined in Sect. 3.2, corresponding to the eigenvalues $s_k$. We can show (Kobayashi and Ren[18])

$$\tilde{\boldsymbol{U}}_k=(\mathcal{Q}^{-1})^2\tilde{\boldsymbol{V}}_k \tag{108}$$

and

$$\tilde{\boldsymbol{U}}'_k\mathcal{D}\tilde{\boldsymbol{V}}_{k'}=\delta_{kk'}. \tag{109}$$

The corresponding projection operator $\tilde{\mathcal{E}}_k$ is given by

$$\tilde{\mathcal{E}}_k=\tilde{\boldsymbol{V}}_k\tilde{\boldsymbol{U}}'_k \tag{110}$$

Therefore, alternative to Eq. (105), we find the following expansion:

$$\boldsymbol{P}^{**}(s, u)=\sum_k\frac{\tilde{\mathcal{E}}_k}{u-u_k}[\boldsymbol{P}^{*}(0, u)+\mathcal{D}\ \boldsymbol{P}^{*}(s,0)] \tag{111}$$

Then by taking the inverse Laplace transform with respect to the variable $u$, we have

$$P^*(s, x) = \sum_k \tilde{\mathcal{E}}_k \Big[ \exp\{u_k x\} \mathcal{D}\, P^*(s, 0)\, U(x) - \frac{\{1 - e^{u_k (x - x_0)}\}}{u_k} P(0, +\infty) \cdot U(x - x_0) \Big] \quad (112)$$

where $U(x)$ is the unit step function.

The eigenvalues $u_k$ 's have the following properties (see Kobayashi and Ren[18] for the proof)

- There total number of eigenvalues is $\prod_{m=1}^{M} (N_m + 1)$.
- The number of nonnegative eigenvalues (i.e., $\mathrm{Re}\{u_k\} > 0$ for $\mathrm{Re}\{s\} > 0$) is equal to the number of integer vector $\boldsymbol{k}$'s that satisfy

$$\sum_{m=1}^{M} R_m k_m < C. \quad (113)$$

We denote these eigenvalues by $u_k^+$.

When the number of "on" sources is such that $\sum_{m=1}^{M} R_m J_m(t) > C$, the buffer cannot be empty at time $t$. Therefore,

$$P_j(t, 0) = 0 \quad (114)$$

for such $\boldsymbol{j}$ that $\sum_{m=1}^{M} R_m j_m > C$. Hence, the number of non-zero functions $\{P_j^*(s, 0)\}$ in $P^*(s, 0)$ is equal to the number of integer vectors $j$'s that satisfy

$$\sum_{m=1}^{M} R_m j_m < C \quad (115)$$

Therefore, from Eqs. (111) and (112) and by extending the analysis in Ren and Kobayashi[26], the unknown boundary condition $P^*(s, 0)$ can be uniquely determined by the following linear constraint equations

- Infinite Buffer Case

$$\tilde{U}'_k \Big[ \frac{e^{-u_k^+ x_0}}{u_k^+} P(0, +\infty) + \mathcal{D}\, P^*(s, 0) \Big] = 0 \quad (116)$$

for each nonnegative eigenvalue $u_k^+$. The number of constraint equations is the number of $\boldsymbol{k}$'s that satisfy Eq. (113).

- Finite Buffer Case

$$\sum_k \tilde{V}_{kj} \tilde{U}'_k \exp\{u_k X\} \Big[ \frac{e^{-u_k x_0}}{u_k} P(0, +\infty) + \mathcal{D}\, P^*(s, 0) \Big] = 0 \quad (117)$$

where $X$ represents the buffer capacity, and $\tilde{V}_{kj}$ is the $\boldsymbol{j}$-th entry of vector $\tilde{\boldsymbol{V}}_k$, which has dimension $L$ and whose entries are lexicographically ordered by the $M$ dimensional vector $\boldsymbol{j}$. The number of constraint equations is equal to the number of $\boldsymbol{j}$'s that satisfy Eq. (115).

See Kobayashi and Ren[18] for an illustrative case, where $N=2$, with $N_1=1$ and $N_2=1$.

## 4. Conclusion and Further Discussion

In this paper we developed a general theory related to the transient analysis of teletraffic, especially the characterization of the so-called *on-off* source models, and the statistical multiplexer for multiple types of traffic. First we reviewed several mathematical techniques: the $z$-transform, matrix representation, spectral analysis, and the Laplace transform. We clarified the relationship between these different methods.

In dealing with the multiple type case, we showed how the *linear operator* interpretation of the partial differential equation in the $z$-transform domain leads to a natural matrix representation, involving the Kronecker sums and products of the components matrices and vectors. We then derived general exact expressions for the non-stationary behavior of the joint distribution of $\boldsymbol{j}$, the vector variable representing the numbers of *on* sources of different types, and $x$, the buffer content variable, as a function of time $t$. We also derived two different spectral expansion representations: one in terms of the eigenvalues $u_k$ 's, the other in terms of the eigenvalues $s_k$ 's.

The general results we have obtained are given in terms of the Laplace transforms, and the joint probability distribution must be obtained by using the numerical inversion method. Ren and Kobayashi[18] report some preliminary results on numerical examples for a single type traffic model.

Once the joint distribution is obtained, a number of performance measures are directly obtainable. Among them is the probability of cell (or packet) blocking, which is of significant importance in designing a multi-media high-speed networks. A cell loss occurs since the multiplexer capacity is, in reality, finite. The cell loss probability due to buffer overflows is often approximated by computing the tail end of the marginal distribution of $x$ that exceeds the *buffer capacity*. In this paper, however, we obtained an exact solution of the probability distribution for a finite-capacity model as well as for the infinite-capacity model.

From an application viewpoint, it is important to address the computational complexity aspect of this result. Certainly computational complexity grows exponentially as the size of the problem becomes large, but in practice it should be sufficient to compute the first few dominant exponential terms in Eq. (112) that correspond to those negative eigenvalues $u_k$ which are close to zero. Kobayashi[17] discusses a simple method to identify *dominant types* of traffic that provide a tight bound on the probability of cell blocking due to buffer overflows. His analysis was done, however, for the steady-states solution in multiple types of traffic source, where each type is represented by the infinite source model, as discussed at the end of Sect. 2.2. Such analysis should be extended to the time-dependent behavior with finite sources, as well.

The multiplexer transient analysis presented in this paper should be useful in formulating flow control models, such as *admission control*. In a high-speed network, the conventional feed-back control scheme based on the *steady state* analysis will fail, since by the time some information on traffic congestion (such as cell blocking at the multiplexer level or call blocking at the network level) is detected and sent to the originating sources, it will be too late for the network to take corrective actions. Thus it is clear that some type of *predictive* control must be formulated, and our result of transient analysis is expected to be valuable in developing such design and control procedures.

There are a few other important areas for further investigations: One such area is to generalize the traffic source model. Elwalid, Mitra and Stern[6], and Stern and Elwalid[28] discuss a Markov model with many states. Zhang[32] discusses some general properties of the time-dependent solution for such a traffic source model.

In order to cope with the computational complexity issue, a general theory of asymptotic approximations and bounding arguments should be explored. Alternatively we may be able to formulate the multiplexer model based on the diffusion approximation as has been discussed by Kobayashi[14] in dealing with multiple access protocols.

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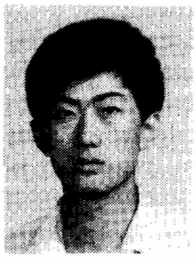

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