A SPECTRAL REPRESENTATION APPROACH TO STATISTICAL MULTIPLEXING OF MULTIPLE TYPES OF TRAFFIC

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Abstract: The cell loss probability is a major performance factor in designing an ATM (asynchronous transfer mode) network for broadband integrated services of multi-media communications. We analyze the buffer overflow probability of a statistical multiplexer with *multiple* types of traffic. An asymptotically tight lower bound of the cell loss probability is obtained by taking a spectral representation approach to the governing differential equation. The exponent term of the bound is characterized by simple parameters of what we term the "dominant" type traffic.

Extended Summary

We assume that there are M types of sources, and the traffic of type m is characterized by the arrival of "bursts" with Poisson rate λ_m. The burst is exponentially distributed with mean $1/\beta_m$, and each burst generates cells at the rate of R_m [cells/sec]. Let $J_m(t)$ be the number of type m bursts at time t. The aggregate rate of cell arrivals at the multiplexer is then given by

$$R(t) = \sum_{m=1}^{M} R_m J_m(t) .$$

When $R(t)$ exceeds C [cells/sec], the output link capacity, all the cells cannot be handled immediately. Let $Q(t)$ denote the number of cells outstanding in the output buffer, and define

$$P(\mathbf{j}, y) = \lim_{t\to\infty} Pr\{J_m(t) = j_m , \; 1 \le m \le M ; \text{ and } Q(t) \le y\} . \quad (1)$$

Following [1] we can derive a differential equation for $P(\mathbf{j}, y)$. We take the multidimensional z-transform of Eq. (1).

$$G(\mathbf{z}, y) = \sum_{j_1} \cdot \sum_{j_2} \cdots \sum_{j_m} P(\mathbf{j}, y) z_1^{j_1} z_2^{j_2} \ldots z_M^{j_M} . \quad (2)$$

We then take the Laplace-Stieltjes transform with respect to the the variable y

$$G^*(\mathbf{z}, s) = \int_0^\infty e^{-sy} G(\mathbf{z}, dy) \quad (3)$$

Then the differential equation for $P(\mathbf{j}, y)$ can be transformed into the following equation for $G^*(\mathbf{z}, y)$ [2].

$$\begin{aligned}\sum_{m=1}^{M} \{sR_m &+ \beta_m(z_m - 1)\} \frac{\partial G^*(\mathbf{z}, s)}{\partial z_m} \\ &= \{sC + \sum_{m=1}^{M} \lambda_m(z_m - 1)\} G^*(\mathbf{z}, s) \\ &+ \sum_{m=1}^{M} (z_m - 1) \left\{ \lambda_m G(\mathbf{z}, 0) - \beta_m \frac{\partial G(\mathbf{z}, 0)}{\partial z_m} \right\} \quad (4)\end{aligned}$$

The second term in the right-hand side of Eq. (4) represents the (unknown) boundary condition. The boundary value $G(\mathbf{z}, 0)$ at $y = 0$ does not affect the solution at large y. If we ignore the boundary condition, the resultant homogeneous linear differential equation will reduce (see [2]) to

$$s\{F(s) - C\} = 0 \quad (5)$$

where

$$F(s) = \sum_{m=1}^{M} \left\{ k_m(R_m + \frac{\beta_m}{s}) + \frac{\lambda_m R_m}{sR_m + \beta_m} \right\} . \quad (6)$$

Here k_m's are the components of vector $\mathbf{k}$ and are all nonnegative integers. For a given integer vector $\mathbf{k}$, there are $(M+1)$ discrete spectral values that s can take on.

Let $s_1(\mathbf{k})$ denote the largest negative spectrum among the roots of Eq. (5). The spectrum $s_1(\mathbf{k})$ takes on its maximum possible value when $\mathbf{k} = \mathbf{0}$. This value $s_1(\mathbf{0})$ is of special importance because it determines the most dominant exponential term of the buffer overflow probability: The probability that the buffer content Q exceeds some predetermined buffer capacity B(cells) is approximately given, for larger B, by

$$Pr\{Q \ge B\} \cong b \exp\{s_1(\mathbf{0})B\} \quad (7)$$

where $b > 0$ and $s_1(0) < 0$.

The numerical evaluation of this largest spectrum $s_1(\mathbf{0})$ can be achieved by any iterative method since we can show

$$\max_m \left\{ -\frac{\beta_m}{R_m} + \frac{\lambda_m}{C} \right\} \le s_1(0) < 0 \quad (8)$$

and the function $F(s)$ defined by (6) is a monotone decreasing function in the interval $[-\frac{\beta_m}{R_m}, 0]$ which includes the range (8). It is instructive to note that the term

$$-\frac{\beta_m}{R_m} + \frac{\lambda_m}{C} = -\frac{\beta_m}{R_m}(1 - \frac{\lambda_m R_m}{\beta_m C}) \quad (9)$$

corresponds to the largest negative spectrum that would be obtained, when all the traffic sources were of type m

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References

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