Asymptotic Non-stationary Behavior of Statistical Multiplexing with Multiple Types of Traffic

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Abstract

The cell loss probability is a major performance factor in designing an ATM (asynchronous transfer mode) network for broadband integrated services of multi-media communications. The dynamic behavior of an ATM network needs to be well understood because of its extremely high speed and diversity of traffic. We analyze the transient buffer overflow probability of a statistical multiplexer with *multiple* types of traffic by taking a spectral representation approach. The joint distribution is obtained in the Laplace transform domain in analytic form. An asymptotic behavior is characterized by simple parameters of what we term the "dominant" type traffic.

Summary

We assume that there are M types of sources, and the traffic of type m is characterized by the arrival of "bursts" with Poisson rate λ_m. The burst length is exponentially distributed with mean $\frac{1}{\beta_m}$, and each burst generates cells at the rate of R_m [cells/sec]. The output link capacity is denoted by C [cells/sec]. To make the system stable, we require $\sum_{m=1}^{M} \frac{\lambda_m R_m}{\beta_m} < C$.

Let $J_m(t)$ be the number of type m burst at time t. The aggregate rate of cell arrivals at the multiplexer is then given by $R(t) = \sum_{m=1}^{M} R_m J_m(t)$. When $R(t)$ exceeds C [cells/sec], all the cells cannot be handled immediately. Let $Q(t)$ denote the number of cells outstanding in the output buffer, and define

$$P_{\mathbf{j}}(t,x) \;=\; Prob\{J_m(t) = j_m,\ 1 \le m \le M;\ \text{and } Q(t) \le x\}. \qquad (1)$$

Let $\mathbf{P}(t,x)$ be the column vector that consists of all the $P_{\mathbf{j}}(t,x)$. Following [1], we can derive a matrix differential equation for $\mathbf{P}(t,x)$:

$$\frac{\partial \mathbf{P}(t,x)}{\partial t} + \mathcal{D}\frac{\partial \mathbf{P}(t,x)}{\partial x} = \mathcal{M}\mathbf{P}(t,x) \qquad (2)$$

where

$$\begin{aligned} \mathcal{M} &= \mathcal{M}_1 \oplus \mathcal{M}_2 \oplus \cdots \oplus \mathcal{M}_M \\ \mathcal{D} &= \mathcal{R}^{(1)} \oplus \mathcal{R}^{(2)} \oplus \cdots \oplus \mathcal{R}^{(M)} - C \cdot I. \end{aligned}$$

Here $\otimes$ and $\oplus$ represent Kronecker product and Kronecker sum, respectively, and I is the identity matrix of infinite dimension. And

$$\mathcal{M}_m = \begin{bmatrix} -\lambda_m & \beta_m & 0 & \cdots \\ \lambda_m & -(\lambda_m+\beta_m) & 2\beta_m & \cdots \\ 0 & \lambda_m & -(\lambda_m+2\beta_m) & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

$$\mathcal{R}^{(m)} = \mathrm{diag}[0, R_m, \cdots, j_m R_m, \cdots].$$

In order to solve Eq.(2), we first take the double Laplace transform $(t,x) \leftrightarrow (s,u)$ on $\mathbf{P}(t,x)$, i.e., $\mathbf{P}(t,x) \leftrightarrow \mathbf{P}^{**}(s,u)$, and use $\mathbf{P}^*(s,0)$ and $\mathbf{P}^*(0,u)$ to denote the Laplace transforms of $\mathbf{P}(t,0)$ and $\mathbf{P}(0,x)$, respectively. Equation (2) then becomes

$$\mathbf{P}^{**}(s,u) = (u\mathcal{D} + sI - \mathcal{M})^{-1}[\mathbf{P}^*(0,u) + \mathcal{D}\mathbf{P}^*(s,0)]. \qquad (3)$$

Let us solve the eigenvalues with respect to u, i.e., $u\mathcal{D}V(s) = (\mathcal{M} - sI)V(s)$, and let $V(s)$ and $V(\mathbf{z};s)$ be the corresponding right eigenvector and its generating function. we assume that $V(\mathbf{z};s)$ can be decomposed as $V(\mathbf{z};s) = \prod_{m=1}^{M} V_m(z_m;s)$, then it follows

$$\begin{aligned} &\textstyle\sum_{m=1}^{M}[(uR_m+\beta_m)z_m - \beta_m]\frac{\partial}{\partial z_m}\{lnV_m(z_m;s)\} \\ =\ & -s + uC + \textstyle\sum_{m=1}^{M}\lambda_m(z_m-1). \end{aligned} \qquad (4)$$

The solution of Eq.(4) should have the following form:

$$V_{\mathbf{k},m}(z_m;s) = exp\{\frac{\lambda_m z_m}{u(s)R_m + \beta_m}\}[\beta_m - (u(s)R_m + \beta_m)z_m]^{k_m}, \qquad (5)$$

where k_m is the mth element of an integer vector $\mathbf{k} = [k_1, \cdots, k_M]$: $k_m = 0, 1, 2, \cdots$ and $m = 1, 2, \cdots, M$.

The eigenvalue $u(s)$, involved in the above equation, is given by solving

$$\sum_{m=1}^{M} \frac{k_m(uR_m+\beta_m)^2 + u\lambda_m R_m}{uR_m + \beta_m} = uC - s. \qquad (6)$$

If we denote $u_{\mathbf{k}}(s)$ the eigenvalue for the integer vector $\mathbf{k}$, and let $V_{\mathbf{k}}(s)$ and $U_{\mathbf{k}}(s)$ be the corresponding right and left eigenvectors, respectively, then they can be represented as

$$\begin{aligned} V_{\mathbf{k}}(s) &= V_{k_1}(s) \otimes \cdots \otimes V_{k_M}(s) \\ U_{\mathbf{k}}(s) &= (\mathcal{Q}^{-1})^2 V_{\mathbf{k}}(s) \end{aligned}$$

with

$$U_{\mathbf{l}}'(s)\mathcal{D}V_{\mathbf{k}}(s) = \delta_{\mathbf{lk}} \qquad (7)$$

and $\mathcal{Q}$ is a diagonal matrix of infinite dimension given by

$$\mathcal{Q} = \mathcal{Q}_1 \otimes \cdots \otimes \mathcal{Q}_M, \quad \mathcal{Q}_m = \mathrm{diag}[1, \cdots, \sqrt{(\tfrac{\lambda_m}{\beta_m})^j \tfrac{1}{j!}}, \cdots]$$

It can be shown from [4] that the number of the positive eigenvalues, denoted by $u_{\mathbf{k}}^+(s)$ and derived from Eq.(6), is equal to the number of $\mathbf{k}$'s that satisfy

$$\sum_{k=1}^{M} R_m k_m < C \qquad (8)$$

Thus, the unknown transient boundary condition $\mathbf{P}^*(s,0)$ can be determined by a set of linear constraint equations

$$U_{\mathbf{k}}'(s)[\mathbf{P}^*(0, u_{\mathbf{k}}^+(s)) + \mathcal{D}\mathbf{P}^*(s,0)] = 0 \qquad (9)$$

Since the dimension of this matrix equation is infinite, we should be interested in the most dominant (largest negative) eigenvalue $u_{dom}(s)$, which is obtained by setting $\mathbf{k} = \mathbf{0}$ in Eq.(6). This dominant term will be of practical importance when we consider an asymptotic buffer behavior, i.e., when x is large enough. The transient probability that the buffer content $Q(t)$ exceeds some predetermined buffer capacity B [cells] is approximately given, for larger B, by

$$G^*(s,B) \stackrel{\text{def}}{=} \mathcal{L}_t\{Prob\{Q(t) > B\}\} \approx b(s)exp\{u_{dom}(s)B\} \qquad (10)$$

where $b(s) = -U_0'(s)[\mathbf{P}^*(0, u_{dom}(s)) + \mathcal{D}\mathbf{P}^*(s,0)]V_0(1;s)$.

We can show that the dominant eigenvalue $u_{dom}(s)$ lies between $\max_m\{-\frac{\beta_m}{R_m}\}$ and 0 for all $s \ge 0$.

References

[1] Kobayashi, H. and Q. Ren [1992], "Non-stationary Behavior of Statistical Multiplexing for Multiple Types of Traffic", *Proc. Twenty-Sixth Annual Conference on Information Sciences and Systems*, Princeton University, Princeton, N.J., March 18-20, 1992.

[2] Kobayashi, H. [1991], "A Spectral Representation Approach to Statistical Multiplexing of Multiple Types of Traffic", *Proc. 1991 IEEE International IT Symposium* , p.156, Budapest, Hungary.

[3] Kosten, L. [1984], "Stochastic Theory of Data handling Systems with Group of Multiple Sources". In H. Rudin and W. Bux (eds.), *Performance of Computer-Communication Systems*, pp. 321-331. North-Holland Publishing Co.

[4] Lancaster, P. and M. Tismenetsky [1985], *The Theory of Matrices*, Academic Press.