Transient Solutions for the Buffer Behavior in Statistical Multiplexing *

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Abstract

We present time-dependent (or transient) solutions for a mathematical model of statistical multiplexer. The problem is motivated by the need to better understand the performance of fast packet switching in asynchronous transfer mode (ATM), which will be adopted in the broadband ISDN. The transient solutions will be of critical value in understanding dynamic behavior of the multiplexer, and loss probabilities at the cell (or packet) level.

We use the double Laplace transforms method, and reduce the partial differential equation that governs the multiplexer behavior to the eigenvalue problem of a matrix equation in the Laplace domain. We derive important properties of these eigenvalues, by extending earlier results discussed by Anick, Mitra and Sondhi [1982] for the equilibrium solutions.

A most critical step in our analysis is to identify a set of linear equations that uniquely determine the time-dependent probability distributions at the buffer boundaries. These boundary conditions are in turn used to solve the general transient solutions. For the infinite buffer case, we show that a closed form solution is given in term of explicitly identified eigenvalues and eigenvectors. When the buffer capacity is finite, the determination of boundary conditions requires us to solve a matrix equation.

We present some numerical results to illustrate our solution technique. A potential application of the time-dependent solution is in the area of preventive congestion control in a high speed network.

1 Introduction

There have been a number of studies reported on queueing theoretic models of statistical multiplexing. As for earlier results, the reader is referred, for example, to Hayes [1984] and Kobayashi and Konheim [1977]. Fluid flow model for the buffer behavior analysis, which can be viewed as a generalization of the birth-and-death process model, has been discussed by Hashida and Fujiki [1973], Cohen [1974], Kosten [1974,1984], Anick, Mitra and Sondhi [1982], Mitra [1988], Morrison [1989], Kobayashi [1990], Elwalid, Mitra and Stern [1991], Coffman, Igelnik and Kogan [1991], and others. This class of probabilistic models, often referred to as *data handling systems* in deference of Kosten's series of papers on the subject, is recognized increasingly important, because it provides a practical mathematical framework to analyze the buffer behavior of statistical multiplexing, which is the basic principle of fast packet switching or ATM

(asynchronous transfer mode) switching in the B-ISDN (broadband integrated services digital network) environment.

All of the above studies, however, have exclusively dealt with the steady-state solutions of the fluid model, whereas the present work is, to the best of the authors' knowledge, a first result on time-dependent solutions.

The dynamics of this statistical multiplexer is characterized by a linear partial differential equation. Although the initial condition is always clearly known, the transient behavior at the buffer boundaries are unknown functions of time. Anick, Mitra and Sondhi [1982] and Mitra [1988] presented the elegent ways to treat the boundary conditions for the steady-state case solution. By generalizing their approach, we develop a procedure to determine the time-dependent boundary functions.

2 Analysis of a Fluid Flow Model

We assume that there are N statistically independent and identical sources and each source alternates between the *on* (or *burst*) state and the *off* (or *silence*) state. We also assume that successive *burst* and *silence* periods of a source form an alternating renewal process, and their durations are exponentially distributed with mean β^{-1} and α^{-1}, respectively. If a source is in the on-state, it will generate packets (or *cells* in the ATM terminology) and its rate is assumed, without loss of generality, to be one packet per unit second. While a source is in the off-state, it generates no packet.

Let C [packets/sec.] denote the multiplexer's output *link capacity*, and $Q(t)$ the total amount of packets outstanding at the multiplexer output link at time t. Let $J(t)$ denote the number of sources in the on-state at time t (see Figure 1). Then, while $J(t) < C$, all arriving packets are transmitted immediately over the output link, thus there will be no packets left in the output link buffer, whereas in the period when $J(t)$ exceeds C, a queue will develop at the rate of $J(t) - C$. Although $Q(t)$ is, strictly speaking, an integer-valued function, we approximate it by a time-continuous function, assuming that a series of packets arrive as a continuous stream of bits. In other words, we represent the stream of packets as a time-continuous fluid flow:

$$\frac{dQ(t)}{dt} = \begin{cases} J(t) - C & \text{if } Q(t) > 0 \text{ or } J(t) > C, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

A typical behavior of the random process $J(t)$ and the corresponding $Q(t)$ are depicted in Figure 2.

We assume here that the buffer capacity is either infinite or finite with upper limit X, and the following stability conditon is satisfied:

$$\rho \overset{\text{def}}{=} \frac{N\alpha}{C(\alpha + \beta)} < 1, \tag{2}$$

where ρ is the traffic intensity.

In order to analyze the behavior of the buffer content $Q(t)$, we consider the *pair process* $\{J(t), Q(t)\}$ and define

$$P_j(t, x) \stackrel{\text{def}}{=} Prob[J(t) = j, Q(t) \leq x], \tag{3}$$

for $0 \leq j \leq N, t \geq 0, x \geq 0$.

By extending the birth-and-death process model equation (see, for example, Kobayashi [1978] p.130), we can derive the following set of equations that the joint probability function $P_j(t, x)$ must satisfy:

$$\begin{aligned}
\frac{\partial P_j(t, x)}{\partial t} &+ (j - C)\frac{\partial P_j(t, x)}{\partial x} \\
= &-[\lambda(j) + \mu(j)]P_j(t, x) + \lambda(j-1)P_{j-1}(t, x) \\
&+ \mu(j+1)P_{j+1}(t, x), \text{ for } 0 \leq j \leq N, x > 0,
\end{aligned} \tag{4}$$

with

$$P_{-1}(t, x) = P_{N+1}(t, x) = 0, \text{ for all } t \text{ and } x.$$

In our multiplexer model, the birth and death rates are given by

$$\lambda(j) = (N - j)\alpha, \quad \mu(j) = j\beta. \tag{5}$$

Writing Eq.(4) in matrix form, we obtain:

$$\frac{\partial P(t, x)}{\partial t} + \mathcal{D}\frac{\partial P(t, x)}{\partial x} = \mathcal{M}P(t, x), \tag{6}$$

where

$$P(t, x) = [P_0(t, x), P_1(t, x), \cdots, P_N(t, x)]^T, \tag{7}$$

and $\mathcal{D}$ is an $(N + 1) \times (N + 1)$ diagonal matrix defined by

$$\mathcal{D} = diag[-C, 1 - C, \cdots, j - C, \cdots, N - C]. \tag{8}$$

We assume that the *link capacity* C is a non-integer number between 0 and N, and satisfies the stability condition Eq.(2).
The matrix $\mathcal{M}$, an $(N + 1) \times (N + 1)$ tridiagonal matrix, is an infinitesimal generator of the Markov process $J(t)$.

3 Transient Analysis:

3.1 Double Laplace Transforms:

Let us first apply the Laplace transforms defined as follows

$$P^*(s, x) = \mathcal{L}_t\{P(t, x)\} = \int_0^{+\infty} e^{-st}P(t, x)dt, \tag{9}$$

$$P^*(t, u) = \mathcal{L}_x\{P(t, x)\} = \int_0^{+\infty} e^{-ux}P(t, x)dx, \tag{10}$$

$$P^{**}(s, u) = \mathcal{L}_x\{P^*(s, x)\} = \int_0^{+\infty} e^{-ux}P^*(s, x)dx. \tag{11}$$

Then Eq.(6) will be transformed into the following matrix equation:

$$(u\mathcal{D} + sI - \mathcal{M})P^{**}(s, u) = P^*(0, u) + \mathcal{D}P^*(s, 0), \tag{12}$$

where I is the $(N + 1) \times (N + 1)$ identity matrix.

The boundary value $P^*(s, 0)$, which characterizes the time dependent behavior of the empty buffer case ($x = 0$), is unknown, but will be determined later. On the other hand, we assume, without loss of generality, that the buffer is empty at $t = 0$. Thus the initial condition $P^*(0, u)$ is given:

$$P^*(0, u) = \mathcal{L}_x\{P(0, x)\} = \begin{bmatrix} \frac{1}{u} \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \tag{13}$$

Therefore Eq.(12) yields

$$\begin{aligned}
P^{**}(s, u) &= [u\mathcal{D} + sI - \mathcal{M}]^{-1}[P^*(0, u) + \mathcal{D}P^*(s, 0)] \\
&= \frac{\mathcal{A}(s, u)}{C(s, u)}B(s, u),
\end{aligned} \tag{14}$$

where

$\mathcal{A}(s, u)$ is the $(N+1) \times (N+1)$ adjugate matrix of $[u\mathcal{D}+sI-\mathcal{M}]$, in which each of its element is a polynomial in both s and u with degrees up to N.

$B(s, u)$, which is $P^*(0, u) + \mathcal{D}P^*(s, 0)$, is an $(N+1) \times 1$ column vector.

$C(s, u)$, the determinant of $[u\mathcal{D} + sI - \mathcal{M}]$, is a polynomial of degree $(N + 1)$ in both u and s.

Let $u_0(s), u_1(s), \cdots, u_N(s)$ be the $N + 1$ roots of the characteristic equation $C(s, u) = 0$.

Note that $B(s, u)$ contains a factor u in its denominator, as given by Eq.(13). Thus the denominator of Eq.(14) is an $(N + 2)$-degree polynomial of u: the first $(N + 1)$ roots are the characteristic roots defined above, and the $(N + 2)$-th root is

$$u_{N+1}(s) = 0 \tag{15}$$

We can show (see Appendix A) that these $(N+2)$ roots are distinct.

By partial fraction, we can write Eq.(14) as

$$P^{**}(s, u) = \frac{H_{N+1}(s)}{u} + \sum_{k=0}^{N} \frac{H_k(s)}{u - u_k(s)}, \tag{16}$$

where

$$H_k(s) = \lim_{u \to u_k(s)}[u - u_k(s)]P^{**}(s, u), \quad 0 \leq k \leq N + 1. \tag{17}$$

By taking the inverse Laplace transform of Eq.(16), we obtain

$$P^*(s, x) = H_{N+1}(s) + \sum_{k=0}^{N} H_k(s)e^{u_k(s)x}. \tag{18}$$

3.2 Properties of Characteristic Eigenvalues and Eigenvectors

The problem of finding the roots of $C(s, u)$ can be reduced to the one of finding eigenvalues of the following matrix equation:

$$u\mathcal{D}V(s) = [\mathcal{M} - sI]V(s), \tag{19}$$

where u is an eigenvalue of $\mathcal{D}^{-1}[\mathcal{M}-sI]$, and $V(s)$ be the associated right eigenvector, whose j-th element is denoted by $V_j(s)$, $0 \leq j \leq N$.

Then the generating function defined by

$$V(z, s) = \sum_{j=0}^{N} V_j(s)z^j \tag{20}$$

satisfies

$$\begin{aligned}
\frac{\partial}{\partial z} \ln V(z, s) &= \frac{uC + N\alpha(z - 1) - s}{\alpha z^2 + (u + \beta - \alpha)z - \beta} \\
&= \frac{K}{z - z_1} + \frac{N - K}{z - z_2},
\end{aligned} \tag{21}$$

which is equivalent (except for a scaling constant) to

$$V(z, s) = (z - z_1)^K(z - z_2)^{N-K}, \tag{22}$$

where

$$z_1, z_2 = \frac{-(u + \beta - \alpha) \pm \sqrt{(u + \beta - \alpha)^2 + 4\alpha\beta}}{2\alpha} \tag{23}$$

and

$$K = \frac{uC + N\alpha(z_1 - 1) - s}{\alpha(z_1 - z_2)}. \tag{24}$$

Observe that by its definition of Eq.(20), $V(z,s)$ is a polynomial in z of degree N. Since z_1 and z_2 are distinct, this is possible if and only if K is an integer between 0 and N.

Then for a given integer K ($0 \le K \le N$), the Eq.(24) yields, after some algebraic manipulation, the following two roots:

$$u_1(K,s), u_2(K,s) \tag{25}$$
$$= \alpha - \beta$$
$$+ \frac{(\frac{N}{2} - C)\{s + (N-C)\alpha + C\beta\} \pm |\frac{N}{2} - K|\sqrt{\Delta(K,s)}}{(\frac{N}{2} - K)^2 - (\frac{N}{2} - C)^2},$$

where

$$\Delta(K,s) = \{s + (N-C)\alpha + C\beta\}^2 \tag{26}$$
$$+ 4\alpha\beta(K-C)(N-K-C)$$

We now obtain the following theorem, which is a generalization of the main theorem in Anick et al. [1982].

Theorem 1:

1. As K varies from 0 to N, Eq.(25) produces a total of $N+1$ *non-zero* and *distinct* roots.

2. The $N+1$ roots can be classified as follows:

   - There are $N - \lfloor C \rfloor$ *negative* roots, i.e., $Re\{u_k(s)\} < 0$ for all $Re\{s\} \ge 0$, and denoted by $\{u_k(s) ; 1 \le k \le N - \lfloor C \rfloor \}$. Each $\{u_k(s)\}$ is a *strictly decreasing* function of $s \ge 0$.

   - There are $\lfloor C \rfloor$ *positive* roots, i.e., $Re\{u_k(s)\} > 0$ for all $Re\{s\} \ge 0$, and denoted by $\{u_k(s); N - \lfloor C \rfloor + 1 \le k \le N \}$. Each $\{u_k(s)\}$ is a *strictly increasing* function of all $s \ge 0$.

   - $u_0(s)$, the remaining root, is a *strictly increasing* function of $s \ge 0$ and is positive, i.e., $Re\{u_0(s)\} > 0$ for all $Re\{s\} > 0$, but $u_0(0) = 0$.

The proof of the theorem is given in Appendix A.

For each $u_k(s)$, the corresponding right eigenvector, denoted $V_k(s)$, can be obtained by its generating function Eq.(22), and the corresponding left eigenvector, denoted by $U_k(s)$, is given by

$$U_k(s) = (Q^{-1})^2 V_k(s), \tag{27}$$

where Q is a diagonal matrix that transforms the matrix $\mathcal{M}$ into a symmetric matrix, and the j-th element of Q is given by

$$\sqrt{(\tfrac{\alpha}{\beta})^{j-1} \binom{N}{j-1}}.$$

We normalize these eigenvectors, which are orthogonal to each other, so that

$$U'_k(s) V_l(s) = \delta_{kl}, \tag{28}$$

where δ_{kl} is Kronecker delta. We find (see Friedman [1956]) that the inverse matrix $[uD + sI - \mathcal{M}]^{-1}$ in Eq.(14) can be represented as

$$[uD + sI - \mathcal{M}]^{-1} = \sum_{k=0}^{N} \frac{V_k(s) U'_k(s)}{u - u_k(s)} D^{-1}, \tag{29}$$

or

$$P^{**}(s,u) = \sum_{k=0}^{N} \frac{V_k(s) U'_k(s)}{u - u_k(s)} [P^*(s,0) + D^{-1} P^*(0,u)] \tag{30}$$

3.3 Transient Boundary Conditions

3.3.1 Infinite Buffer Case: $z = 0$

If we assume the buffer capacity to be infinite, the unknown boundary condition $P^*(s,0)$ or $\mathcal{L}_t\{P(t,0)\}$ characterizes the transient behavior of the empty buffer. Although it does not affect the roots $u_k(s)$, it influences $H_k(s)$'s of Eq.(18) as is evident from Eq.(17). Therefore we need to solve $P^*(s,0)$ in conjunction with the stable solution.

From Eq.(18) and *Theorem 1*, we find that the solution is stable, if

$$H_k(s) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = 0, \tag{31}$$

for $k = 0$ and $N - \lfloor C \rfloor + 1 \le k \le N$.

From Eqs.(17) and (14) we see that the above condition is equivalent to

$$\mathcal{A}(s, u_k(s)) \cdot B(s, u_k(s)) = 0 \tag{32}$$

for $k = 0$ and $N - \lfloor C \rfloor + 1 \le k \le N$.

Now we make the following observation regarding the empty buffer, by generalizing the properties discussed by Kosten [1974] and Anick et al. [1982].

When the incoming traffic rate is greater than the network link capacity at any time t, i.e., $J(t) > C$, the buffer content necessarily increases and the buffer cannot stay empty. Thus it follows that,

$$P_j(t,0) = 0, \tag{33}$$

for $\lfloor C \rfloor + 1 \le j \le N$ and for all $t \ge 0$, which also implies

$$P^*(s,0) = [P_0^*(s,0), P_1^*(s,0), \cdots, P_{\lfloor C \rfloor}^*(s,0), 0, 0, \cdots, 0]^T. \tag{34}$$

A total of $\lfloor C \rfloor + 1$ unknown elements $P_0^*(s,0), P_1^*(s,0), \cdots, P_{\lfloor C \rfloor}^*(s,0)$ must now be determined.

We state as a theorem the following property which is critical in determining the unknown boundary conditions $P^*(s,0)$:

Theorem 2 :

For each $u_k(s)$, $k = 0$ or $N - \lfloor C \rfloor + 1 \le k \le N$, Eq.(32) gives exactly *one* (instead of $N+1$) *linear constraint* equation of variables $P_0^*(s,0), P_1^*(s,0), \cdots, P_{\lfloor C \rfloor}^*(s,0)$.

By setting k equal to $0, N - \lfloor C \rfloor + 1, \cdots$, and N, we obtain from Eq.(32) $\lfloor C \rfloor + 1$ *distinct linear constraint* equations, which *uniquely* determine $\lfloor C \rfloor + 1$ unknown variables, i.e., $P_0^*(s,0), P_1^*(s,0), \cdots, P_{\lfloor C \rfloor}^*(s,0)$.

The proof is given in Appendix B.

3.3.2 Finite Buffer Case: $z = 0$ and $z = X$

If we assume the buffer to be finite with its upper limit X, we may not have the constraint equations, analogy to the infinite buffer case, on the unknown boundary conditions $P^*(s,0)$ and $P^*(s,X)$. By generalizing observations made for the steady-state case, i.e., Mitra [1988], we have the following two sets of equations on the buffer boundaries:

1. For $z = 0$:
   When the incoming traffic rate is greater than the network link capacity at any time t, i.e., $J(t) > C$, the buffer cannot stay empty. Thus it follows that

$$P_j^*(s,0) = 0, \tag{35}$$

for $\lfloor C \rfloor + 1 \le j \le N$.

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2. For $x = X$:

When the incoming traffic rate is less tha the network link capacity at any time t, i.e., $J(t) < C$, the buffer cannot stay at its upper limit. Thus it follows that

$$P_j^*(s, X) = h_{N+1,j}(s), \qquad (36)$$

where $0 \le j \le \lfloor C \rfloor$ and $h_{N+1,j}(s)$ is the $(j+1)$-th entry of column vector $\mathbf{H}_{N+1}(s)$.

4 Solutions and Discussion

4.1 Infinite Buffer Case: Closed Form Solution

From Eq.(18) and section 3.3.1, our transient solution is given in the form of Laplace transform with respect to the time domain:

$$
\begin{aligned}
\mathbf{P}^*(s, x) &= \mathcal{L}_t\{\mathbf{P}(t, x)\} \qquad (37)\\
&= \mathbf{H}_{N+1}(s) + \sum_{k=1}^{N-\lfloor C \rfloor} \mathbf{H}_k(s) \cdot e^{u_k(s) \cdot x},
\end{aligned}
$$

Although the above solution requires us to solve a matrix equation as stated in *Theorem 2*, we now derive a closed form solution by extending Anick et al. [1982].

From Eq.(30) we find that $\mathbf{P}^*(s, x)$ can be represented as

$$\mathbf{P}^*(s, x) = \mathbf{H}_{N+1}(s) + \sum_{k=1}^{N-\lfloor C \rfloor} [\mathbf{H}_k^{(1)}(s) + a_k(s)\mathbf{V}_k(s)]e^{u_k(s)x} \quad (38)$$

where

$$
\begin{aligned}
\mathbf{H}_k^{(1)}(s) &= [h_{k,0}^{(1)}(s), \cdots, h_{k,N}^{(1)}(s)]' \\
&= \mathbf{V}_k(s)\mathbf{U}_k'(s)\mathcal{D}^{-1}\mathbf{P}^*(0, u_k(s)) \qquad (39)\\
\mathbf{H}_{N+1}(s) &= [h_{N+1,0}(s), \cdots, h_{N+1,N}(s)]' \\
&= -\sum_{k=0}^{N} \frac{\mathbf{V}_k(s)\mathbf{U}_k'(s)}{u_k(s)}\mathcal{D}^{-1}(u\mathbf{P}^*(0, u)|_{u=0}) \quad (40)\\
a_k(s) &= \mathbf{U}_k'(s)\mathbf{P}^*(s, 0) \qquad (41)
\end{aligned}
$$

Define

$$
\begin{aligned}
c_k(s) &= h_{k,N}^{(1)}(s) + a_k(s)V_{k,N}(s) \qquad (42)\\
\mathbf{c}(s) &= [c_1(s), \cdots, \cdots, c_{N-\lfloor C \rfloor}]' \qquad (43)
\end{aligned}
$$

and observe the last entry of l-th derivatives of $\mathbf{P}^*(s, x)$ at $x = 0$, i.e., $\frac{d^{(l)}\mathbf{P}^*(s,x)}{dx^l}|_{x=0}$, $l = 0, \cdots, N - \lfloor C \rfloor - 1$, we have

$$\mathbf{T}\mathbf{c}(s) = -h_{N+1,N}(s)\mathbf{e}, \qquad (44)$$

where $T_{ij} = (u_k(s))^i$, $i = 0, \cdots, N - \lfloor C \rfloor - 1$, and $\mathbf{e} = [1, 0, \cdots, 0]'$. Note that $\mathbf{T}$ is a Vandermonde matrix, the matrix equation (44), after some algebraic manipulation, gives us

$$c_k(s) = -h_{N+1,N}(s) \prod_{i=1, i \neq k}^{N-\lfloor C \rfloor} \frac{u_i(s)}{u_i(s) - u_k(s)}, \qquad (45)$$

for $1 \le k \le N - \lfloor C \rfloor$.

And then , from Eq.(42)

$$a_k(s) = \frac{c_k(s) - h_{k,N}^{(1)}(s)}{V_{k,N}(s)}. \qquad (46)$$

4.2 Finite Buffer Case

For the finite buffer case, the *positive* eigenvalues, i.e., $\{u_k(s); k = 0, N - \lfloor C \rfloor + 1, \cdots, N\}$ can be allowed. Thus our transient solution

is given from Eq.(14) as:

$$\mathbf{P}^*(s, x) = \mathbf{H}_{N+1}(s) + \sum_{k=0}^{N} \mathbf{H}_k(s)e^{u_k(s)x} \qquad (47)$$

or

$$\mathbf{P}^*(s, x) = \mathbf{H}_{N+1}(s) + \sum_{k=0}^{N} [\mathbf{H}_k^{(1)}(s) + a_k(s)\mathbf{V}_k(s)]e^{u_k(s)x}, \quad (48)$$

where $\mathbf{H}_{N+1}(s)$, $\mathbf{H}_k^{(1)}(s)$ and $a_k(s)$ are the same as those in Eqs. (17), (40) and (41) except that k ranges from 0 to N instead of $1 \le k \le N - \lfloor C \rfloor$.

Note that the two set of equations found in section 3.3.2 gives exactly $N + 1$ linear constraint equations on $\{a_k(s); 0 \le k \le N\}$. Thus, $a_k(s)$ of Eq.(41) for $0 \le k \le N$ can be uniquely determined by solving a matrix equation of dimension $N + 1$.

4.3 Discussion

We have obtained the transient solutions of Eq.(6) for both infinite and finite buffer cases. Our final solution $\mathbf{P}^*(s, x)$ is given in the form of Laplace transform with respect to the time domain. A numerical-inversion method of the Laplace transform must be applied to obtain $\mathbf{P}(t, x) = \mathcal{L}_s^{-1}\{\mathbf{P}^*(s, x)\}$.

For the infinite buffer case, our solution is given in a closed form as evident from Eqs.(38) and (46); for the finite buffer case, the solution is also represented by a closed form Eq.(48) except that a matrix equation needs to be solved as given in section 3.3.2.

Although we assume, for simplicity, that all sources are initially off and the buffer is empty at $t = 0$, our analysis can be easily extended to an arbitrary initial condition. Namely, we can assume that $j_0(0 \le j_0 \le N)$ sources are on and the buffer content $Q(t) = x_0$ at $t = 0$. Essentially the same steps given in Sections 3 , 4.1 and 4.2 carry over, resulting in a slightly complicated versions of the final solutions Eqs.(38) and (48).

5 Numerical Examples

We present here some numerical results for the case of $N = 2$ to illustrate our solution technique.

First we obtain transient boundary condition $\mathbf{P}^*(s, 0)$ by solving the linear equations in Eq.(32). This in turn gives $\mathbf{P}^*(s, x)$ by Eq.(37). We then apply the numerical-inversion method of the Laplace transform to $\mathbf{P}^*(s, x)$ with respect to s (see Kobayashi [1978] pp.73-74), and obtain

$$\mathbf{P}(t, x) \simeq \frac{e^{ct}}{T}[\frac{1}{2}Re\{\mathbf{P}^*(c, x)\} + \sum_{k=1}^{+\infty} Re\{\mathbf{P}^*(c + \frac{\pi ik}{2T}, x)\}cos\frac{\pi kt}{2T}],$$

where T is the finite range over which we wish to evaluate $\mathbf{P}(t, x)$, and c is an arbitrary number in the convergence region of $\mathbf{P}^*(s, x)$.

It should be noticed from Eq.(34) that the time-dependent boundary condition $\mathbf{P}^*(s, 0)$ are different, depending on whether $0 < C < 1$ or $1 < C < 2$. The parameters used here are given by $\alpha^{-1} \approx 650ms$, $\beta^{-1} \approx 352ms$ which correspond to the single-source model used in Sriram et al [1986].

6 Conclusion

We have presented mathematical results on the time-dependent behavior of the fluid flow model of statistical multiplexer. From *Theorem 1*, *Theorem 2* and Section 3.3.2, we can uniquely determine the time-dependent boundary conditions by solving a set of linear equations, and then obtain solutions in the form of Laplace transform. Our final solutions are given in closed form with almost no computational complexities.

By extending our analysis, we should be able to develop a new method to predict the network load in real time. A prediction

model will help us gain some insight into the design and analysis of network congestion control. We expect that in a high-speed network, most existing control strategies will fail due to its large propogation delay as compared with the small transmission time. An accurate prediction of the transient network load will enable us to develop a preventive control of network congestion at the cell level by regulating traffic, or dynamically assigning the link capacity.

Appendix A: Proof of Theorem 1

1. Note that $u_1(K, s), u_2(K, s)$, as given in Eq.(25), depend on K only through $(\frac{N}{2} - K)^2$ or $|\frac{N}{2} - K|$. Thus we need to consider only $0 \leq K \leq \frac{N}{2}$ (Anick et al.[1982]). For each $0 \leq K \leq N$, there are two different roots. In total we have $N + 1$ distinct roots as K varies from 0 to $\lfloor \frac{N}{2} \rfloor$.

   None of the roots is zero because $|sI - \mathcal{M}| \neq 0$ for some s.

2. The region of convergence of Laplace transform is a strip parallel to the imaginary axis in the s-plane. In order to show $Re\{u_k(s)\} > 0$ or $Re\{u_k(s)\} < 0$ in the right half plane, i.e., $Re\{s\} \geq 0$, it is sufficient to consider them along the nonnegative real axis.

   In the case $s = 0$, $u_1(K, 0)$ and $u_2(K, 0)$ ($0 \leq K \leq N$) reduce to those of the steady-state case discussed by Anick et al.[1982].

   - Consider those roots which are negative at $s = 0$. There exist $N - \lfloor C \rfloor$ such roots, which we denote by $u_1(s), \cdots, u_{N-\lfloor C \rfloor}(s)$. Their derivatives $\frac{du_k(s)}{ds}$ are always negative.
   - Consider those roots which are positive at $s = 0$. There exist $\lfloor C \rfloor$ such roots, which we denote by $u_{N-\lfloor C \rfloor+1}(s), \cdots, u_N(s)$. Their derivatives $\frac{du_k(s)}{ds}$ are always positive.
   - The root that corresponds to $K = 0$ intersects the origin at $s = 0$:
     $$u_1(0, 0) = 0. \tag{49}$$
     Let $u_0(s)$ denote this root, and we find $\frac{du_0(s)}{ds} > 0$.

   This concludes the Part 2 of Theorem 1.

Appendix B: Proof of Theorem 2

First we need the following lemma:
Lemma

   For two $N \times N$ square matrices, $\mathcal{A}$ and $\mathcal{B}$ given; if $\mathcal{A} \bullet \mathcal{B} = 0$, then

   $$Rank(\mathcal{A}) + Rank(\mathcal{B}) \leq N.$$

In our case, from the definition of Eq.(14) we have

$$[u\mathcal{D} + sI - \mathcal{M}]\mathcal{A}(s, u) = det|u\mathcal{D} + sI - \mathcal{M}| \cdot I = C(s, u)I.$$

Since each $u_k(s), 0 \leq k \leq N$, is one of the $N + 1$ *distinct* roots of C(s,u), the above equation implies

$$[u_k(s)\mathcal{D} + sI - \mathcal{M}]\mathcal{A}(s, u_k(s)) = 0, \quad 0 \leq k \leq N. \tag{50}$$

We also find

$$Rank[u_k(s)\mathcal{D} + sI - \mathcal{M}] = N, \quad 0 \leq k \leq N. \tag{51}$$

By applying the above lemma, we can claim:

$$Rank[\mathcal{A}(s, u_k(s))] \leq 1, \quad 0 \leq k \leq N.$$

Note that $\mathcal{A}(s, u_k(s))$ is not a zero matrix, thus it follows that

$$Rank[\mathcal{A}(s, u_k(s))] = 1, \quad 0 \leq k \leq N \tag{52}$$

Since $Rank[\mathcal{A}(s, u_k(s))] = 1$, all the rows are a multiple of each other. Thus we need to select any non-zero row vector, say the first row $[A_{00}^k(s), A_{01}^k(s), \cdots, A_{0N}^k(s)]$.

Therefore the matrix equation (32) is equivalent to

$$[A_{00}^k(s), A_{01}^k(s), \cdots, A_{0N}^k(s)] \cdot \begin{bmatrix} \frac{1}{u_k(s)} - CP_0^*(s, 0) \\ (1 - C)P_1^*(s, 0) \\ \vdots \\ (\lfloor C \rfloor - C)P_{\lfloor C \rfloor}^*(s, 0) \\ 0 \\ \vdots \\ 0 \end{bmatrix} = 0, \tag{53}$$

which is a linear constraint equation for $P_0^*(s, 0), P_1^*(s, 0), \cdots, P_{\lfloor C \rfloor}^*(s, 0)$.

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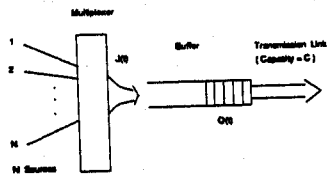
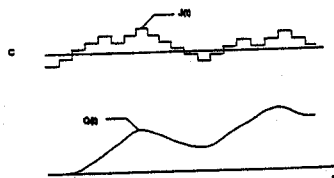Figure 1: The Packet Switching Statistical Multiplexer

Figure 2: The Total Number of Active Sources $J(t)$ and the Number of Queued Packets $Q(t)$.
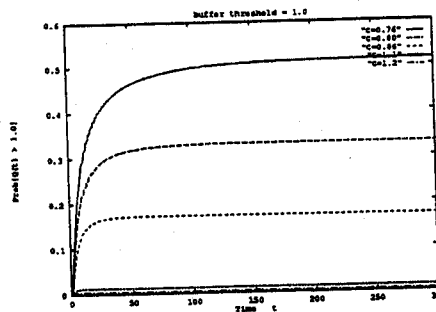

Figure 3: Transient overflow probabilty with $x = 1$. For $C = 0.76, 0.80, 0.86, 1.1$ and 1.2, the traffic intensity ρ is $0.92, 0.88, 0.82, 0.64$ and 0.59, respectively.
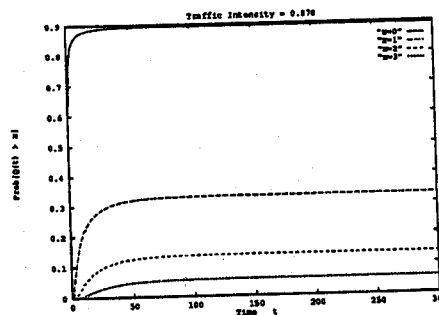

Figure 4: Transient overflow probability with $C = 0.8$ (or $\rho = 0.88$). For $x = 0, 1, 2$ and 3.
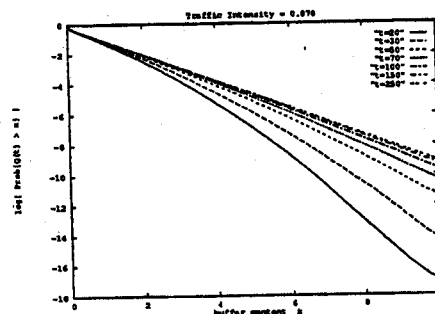

Figure 5: Probabilty of overflow vs buffer size with $C = 0.8$ (or $\rho = 0.878$). At time t is $20, 30, 50, 70, 100, 150$ and 250.
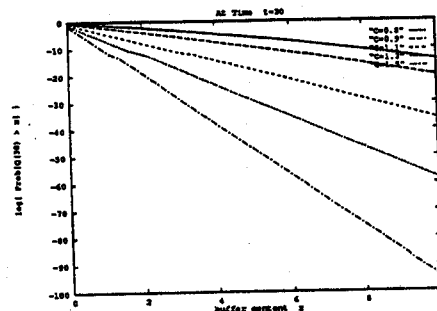

Figure 6: At time $t = 30$, probabilty of overflow vs buffer size. For $C = 0.8, 0.9, 1.1, 1.2$ and 1.5, the traffic intensity ρ is $0.88, 0.78, 0.64, 0.54$ and 0.47, respectively.