FURTHER RESULTS ON A QUEUEING NETWORK MODEL

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ABSTRACT

Recently a "network of queue" representation of a multiple-resource model has been applied to a variety of practical problems such as a multiprogrammed computer model, storage hierarchy model, computer communication network model, etc. These models, however, have always assumed that the routing behavior of each job is governed by a 1st-order Markov chain. Here we eliminate this restriction, and discuss how an analytic solution can still be obtained. We then show that the joint queue-size distribution has a surprisingly simple form under fairly general conditions. The only parameter that enters into the solution is the expectation of the total work placed on each server by a job during its life in the system. The order in which queues are visited is immaterial.

1. INTRODUCTION

A "network of queues" representation of a multiple-resource model plays an important role in the performance analysis of computer/computer-communication systems. Jackson [1] considered a network of exponential servers and showed that for any work-conserving queue discipline the joint queue-size distribution is given in the product form of marginal distributions. Recently, Jackson's model has been extended to the case of multi-class jobs and general service distributions under certain types of queue disciplines [2,3,4]. All these works, however, maintain the assumption that the job transition behavior is governed by a 1st-order Markov chain, i.e., a job completing service at server i will go next to server j with probability p_{ij} of the job's past history. In the present paper we essentially replace this assumption with a Markov chain of arbitrary order.

We shall show that the only quantities which enter into the expression for the queue size distribution are the expectations of the total work demands jobs place on each queue during their stay in the system. In other words, the queue size distribution is robust with respect to all the detail of the routing.

The method to prove this result is to enlarge the state space by the introduction of fictitious classes. In the enlarged state space, the problem falls into the class of models solved in [2].

2. A Cyclic Queue with a General Distribution for the Number of Cycles.

Let us start with the simple queueing network shown in Figure 1. We define a "cycle" to be a routing of a job from the branching point A to server 1, server 0, and back to A. Under the assumption of 1st order Markovian routing, the number of cycles, k, which a given job makes during its lifetime in the system is geometrically distributed, i.e.,

$$p_k = (1-\alpha)\alpha^k, \quad k = 0,1,2,\ldots \tag{1}$$

where α is the probability that a job cycles back to server 1 from the point A.

Consider now a general discrete distribution $\{p_k,\ k\geq 0\}$ for the number of cycles. We assume that p_k have a rational p.g.f. (probability generating function), viz.

$$P(z) = \sum_{k=0}^{\infty} p_k z^k = \frac{R(z)}{Q(z)}. \tag{2}$$

Then we can obtain the following expansion:

$$P(z) = b_0 + \sum_{r=1}^{q} a_0 a_1 \cdots a_{r-1} \cdot b_r \prod_{i=1}^{r} \frac{1-\alpha_i}{1-\alpha_i z} \tag{3}$$

where q is the degree of the polynomial $Q(z)$, and $\{\alpha_i^{-1}\}$ are the characteristic roots of $Q(z)=0$. The representation (4) is schematically shown in Figure 2. It is equivalent to cascaded geometric distributions with parameters $\alpha_1, \alpha_2, \ldots, \alpha_q$. In general, this representation involves the formal use of complex transition probabilities, since the characteristic roots α_i^{-1} of $Q(z)$ can take on complex values [5].

In Figure 2 each box labelled z corresponds to one cycle in Figure 1. Corresponding to the q geometric stages defined above, we now introduce q fictitious classes as shown in Figure 3. A job whose routing is in its rth stage is classified as a class-r job, $1 \leq r \leq q$. For notational convenience we define that all entering jobs proceed first to server 0 as class-0 jobs. A job, after receiving its first service at server 0, either leaves the system immediately, or becomes a class-1 job and cycles around the servers 1 and 0 as many as k_1 times, where the random variable k_1 is geometrically distributed, i.e., $p_{k_1} = (1-\alpha_1)\alpha_1^{k_1}$ $k_1=0,1,2,\ldots$ The job then leaves the system, otherwise changes its status to a class-2 job and cycles around k_2 times, and so forth. Clearly the total number of cycles has the given distribution p_k. The average number of visits a job makes to servers j with class membership r during its life in the system is

$$e_{j,r} = a_0 a_1 \cdots a_{r-1} \frac{\alpha_r}{1-\alpha_r}, \quad j=0,1 \quad 1 \leq r \leq q. \quad (4)$$

For r=0, we set $e_{0,0} = 1$ and $e_{1,0} = 0$.

Now the solution is readily obtained. For simplicity we assume constant arrival rate λ and constant processing rates C_0 and C_1. At each visit a job places a service-demand on the server. Successive service demands be exponentially distributed r.v.'s with mean $\bar{w}_j$, j=0,1. Let $\vec{n} = (n_{01}, n_{02}, \ldots n_{0q}, n_{11}, n_{12}, \ldots n_{1q})$ be the expanded state vector and $\vec{z} = (z_{01}, z_{02}, \ldots z_{0q}, z_{11}, z_{12}, \ldots z_{1q})$ the corresponding vector of transform variables.

With these definitions, the p.g.f. for the queue size distribution [4] becomes

$$G(\vec{z}) = \frac{1-\rho_0}{1-\sum \rho_{0r} z_{0r}} \cdot \frac{1-\rho_1}{1-\sum \rho_{1r} z_{1r}} \quad (5)$$

where $\rho_{0r} = \lambda e_{0r} \bar{w}_0 / C_0$, $\rho_{1r} = \lambda e_{1r} \bar{w}_1 / C_1$, $\rho_0 = \sum \rho_{0r}$, $\rho_1 = \sum \rho_{1r}$ and the sums extend over r=0,1,...q. The marginal distribution of queue j is obtained by substituting $z_{j0} = z_{j2} = \ldots = z_{jq} = z$ and all other z-variables are set to unity. This procedure yields

$$G_j(z) = \frac{1-\rho_j}{1-\rho_j z}, \quad j=0,1 \quad (6)$$

the familiar expression for the M/M/1 queue. We call the quantities ρ_j, j=0,1 <u>workload intensities</u>. They may be expressed in terms of work demand, arrival rate and service rates as follows

$$\rho_j = \sum_{r=0}^{q} \rho_{jr} = \frac{\lambda \bar{w}_j}{C_j} \sum_{r=0}^{q} e_{jr} = \frac{\lambda W_j}{C_j} \quad (7)$$

where $W_j = \bar{w}_j \sum_{r=0}^{q} e_{jr}$. The average number of visits is

$$e_j = \sum_{r=0}^{q} e_{jr} = \begin{cases} 1 + E[k] & \text{for } j=0 \\ E[k] & \text{for } j=1 \end{cases} \quad (8)$$

where E[k] is the average number of cycles. Therefore we may interpret W_j as the expectation of the <u>total work a job places on server j during its stay in the system</u>. Thus only the mean value of the cycle distribution enters into the final result. We say the queue size distribution to be robust with respect to the cycle distribution.

The restrictions placed on arrival and service rates can easily be removed. Our result is compatible with the full generality found in [2-4], viz (1) The arrival rate may depend on the total number of jobs in the system, (2) the processing rate may depend on the local queue size, (3) if the queue discipline is PS (processor-sharing) or preemptive-resume LCFS (last-come, first-served) then a general work demand distribution is permitted.

In the above derivation we assumed that the service demands placed on the server j by a job in its successive cycles are i.i.d. random variables, and hence have the common mean $\bar{w}_j$. By a straightforward extension of the argument, however, we can show that the solution holds for PS and LCFS, even when the service demands are not identically distributed. What counts in the queue size distribution is the total average work a job brings in. How the total work is distributed in individual cycles is immaterial. This is a surprisingly strong result. We have to stress again, however, that for FCFS or any work-conserving discipline other than PS or LCFS the service demands in successive cycles must be drawn from identical exponential distributions.

3. Job Routing Characterized by a High-Order Markov chain

The notion of "cycles" cannot be easily extended to a queueing network with general topology. We will therefore take a different approach in this section: the first-order Markov chain, which characterizes Jackson's model [1] and other related work [2,4], will now be replaced by a high-order Markov chain. Let us start again with a simple network with two servers which we denote as server 0 and server 1. In the 1st-order Markov model, the transition probabilities $p_{s,s'}$ were defined over s,s'=0,1. Now let us assume that job routing is statistically characterized by a <u>2nd-order</u> Markov chain. Then the probabilities $p_{s,s'}$ are now defined over states s,s'=(00), (01), (10), (11).

A job is said to be in state (ij) if the job is now at server j just after completing service at server i. For notational conciseness we use integers 0, 1, 2 and 3 to denote the states (00) (01) (10) and (11), respectively. Therefore, jobs in either 0 or 2 are located at server 0 and those in states 1 or 3 are at server 1. By using the discrete time (or step) parameter k, a job routing can conveniently be represented by a "trellis" picture of Figure 4.

In Figure 4 we introduce an additional state, s=4, which is an absorbing state. A transition to state 4 at step k means that the job leaves the system after k services. For example, the path $0 \to 1 \to 3 \to 2 \to 4$ in Figure 4 means that a job enters server 0 first, moves to server 1, and again to server 1, and finally goes back to server 0 and then leaves the system. Let us denote by $e_s(k)$ the probability that a job is in state s at step k, $\sum_{s=0}^{4} e_s(k) = 1$ for all k. Then the equilibrium state distribution of the extended system state vector

$$\vec{n} = \{n_{s,k};\ 0 \le s \le 3;\ k = 0, 1, 2, \ldots\} \quad (9)$$

is given again by eq. (5).

The quantity e_0 defined by

$$e_0 = \sum_{s=0,2} \sum_{k=0}^{\infty} e_s(k) \quad (10)$$

is the total average number of visits a job makes to server 0 during the job's life time. Similarly, a job visits server 1, on the average, as many as

$$e_1 = \sum_{s=1,3} \sum_{k=0}^{\infty} e_s(k) \quad (11)$$

times. By using $W_j = e_j \bar{w}_j$ we are led again to the simple expression (6).

The evaluation of e_j, j=0,1 can be done by a straightforward application of the Markov chain theory. The probabilities $\{e_s(k)\}$ satisfy the equation

$$e_s(k) = \sum_{s'=0}^{4} e_s(k-1) p_{s's} \quad (12)$$

for $0 \le s \le 4$ and $k \ge 1$. By defining a row vector

$$\underline{e}(k) = [e_0(k), e_1(k), \ldots, e_4(k)], \quad (13)$$

and the corresponding generating function vector

$$\underline{E}(z) = \sum_{k=0}^{\infty} \underline{e}(k) z^k \quad (14)$$

we obtain from (45)

$$\underline{E}(z) - \underline{e}(0) = z\, \underline{E}(z)\ \underline{P} \quad (15)$$

where $\underline{P}$ is the matrix $[p_{s's}]$. We can then derive the well-known formula

$$\underline{E}(z) = \underline{e}(0)\ [\underline{I} - z\underline{P}]^{-1} \quad (16)$$

Denoting the individual components of $\underline{E}(z)$ and $E_s(z)$, $0 \le s \le 4$, we have

$$e_0 = E_0(1) + E_2(1) \quad (17)$$

and

$$e_1 = E_1(1) + E_3(1) \quad (18)$$

4. Extensions to a General Network Topology with Markov Chain of Higher Order

The presentation of Section 3 was made by choosing the simplest example, i.e., a network of two servers and the 2nd-order Markov chain. Its extensions to a queueing network with general topology with job routings characterized by a higher-order Markov chain is now straightforward. If there are m queues 0,1,2,...,m-1 in a network and the job routing transitions are characterized by an hth-order Markov chain, there are m^h different states a job can take on, which we denote as before by integers s=0,1,2, ...m^h-1. We then form a trellis picture with m^h different states plus an absorbing state which we denote by $s=m^h$. We define $e_s(k)$ as before for s=0,1,2,...m^h and k=0,1,2,... We then define parameters

$$e_j = \sum_{\substack{s=j \\ (\mathrm{mod}\ m)}} \sum_{k=0}^{\infty} e_s(k) \quad (19)$$

where the summation over s is taken over those s which satisfy s(modulo m) =j. For example e_0 is obtained by summing over $s=0, m, 2m, \ldots, m^{h-1}$. Similarly, e_1 is the sum of the terms with s=1, m+1, 2m+1,...,m^{h-1} + 1, and so forth. The general form of the solution is given in eq. 20 below. Again only the expected total workdemand $W_j = e_j \bar{w}_j$ enters into the solution. Throughout the above discussion we assumed that the network is open, and thus jobs arrive with rate λ. It is not difficult to extend the result to a closed network as was done in the earlier work [2,3,4]. We define in this case $\lambda=1$. The parameters $\{e_j\}$ are not determinable up to a common scaling factor. If we choose the factor such that $e_{j^*}=1$ for some j*, then $\{e_j\}$ represents the average number of visits that a job makes to server j between its consecutive visits to server j*. The workload parameter W_j then represents the expected total amount of work that the job brings into server j during that cycle.

5. Summary and Conclusion

For a given general network with servers 1, 2,..., m, the distribution of queue-size vector $\vec{n} = (n_1, n_2, \ldots, n_m)$ is given by

$$p(\vec{n}) \begin{cases} = c\, \Lambda(|\vec{n}|) \prod_{j=0}^{m-1} f_j(n_j), & \text{if } \vec{n} \text{ is feasible} \\ = 0, & \text{otherwise,} \end{cases} \quad (20)$$

where c is the normalization constant,

$$|\vec{n}| = \sum_{j=0}^{m-1} n_j \quad (21)$$

$$\Lambda(|\vec{n}|) = \prod_{i=1}^{|\vec{n}|} \lambda(i-1) \quad (22)$$

where $\lambda(N)$ represents the Poisson arrival rate of jobs to the network when the total job population is N. The function $f_j(n_j)$ of (20) is given by

$$f_j(n_j) = D_j(n_j)W_j^{n_j} \quad (23)$$

The function $D_j(n_j)$ is defined by

$$D_j(n_j) = \prod_{n=1}^{n_j} \frac{1}{C_j(n)} \quad (24)$$

where $C_j(n)$ is the processing rate [work units/sec] of server j when its queue size is n, thus called the queue-dependent processing rate function. The parameter W_j [work units/job] of (23) represents the expected work demands that a job places on server j during the entire life of its job.

The most important aspect of this result is its simplicity as well as its generality, namely:

(1) One may characterize the workload of a job in terms of an nth-order Markov chain and service demand distributions. But the only parameter that appears in the final expression for the queue-size distribution is the set of $\{W_j\}$ defined above. In Jackson's model the parameters reduce to

$$W_j = e_j\bar{w}_j, \quad 1 \le j \le m \quad (25)$$

where $\bar{w}_j$ is the average work [work units/visit] at server j, and $\{e_j\}$ is the solution of simultaneous equations

$$e_j = p_{o,j} + \sum_{i=1}^{m} e_i p_{ij}, \quad 1 \le j \le m \quad (26)$$

where $p_{o,j}$ is the probability that a newly-arrived job goes first to server j.

(2) The functional form of equation (23) holds (i) for the service station j with exponential distribution under any work-conserving queue discipline; (ii) for any general distribution if the queue discipline is either processor-sharing or LCFS with preemptive-resume, or the server itself has ample parallel servers (often called infinite servers); and (iii) for different classes of jobs [2,3,4].

(3) An interpretation of equation (20) in the case of a closed network is the same as discussed in earlier work [1-4]. Furthermore, an extension to the case where job routing is characterized by multiple closed chains can be done as discussed in [4]. The efficient computational algorithm [4] to evaluate the normalization constant c of (20) and other related performance measures is readily applicable to our generalized solution as well.

References

[1] J. R. Jackson, "Job Shop-like Queueing Systems," Management Science, vol. 10 (1963) pp. 131-142.

[2] F. Baskett, K. M. Chandy, R. R. Muntz, and F. G. Palacios, "Open, Closed, and Mixed Networks of Queues with Different Classes of Customers," to appear in Journal of the ACM.

[3] H. Kobayashi and M. Reiser, "Some Results on Queueing Models with Different Classes of Customers," Proceedings of the Eighth Annual Princeton Conference on Information Sciences and Systems, March, 1974.

[4] M. Reiser and H. Kobayashi, "Queueing Networks with Multiple Closed Chains" Theory and Computational Algorithms," IBM Research Report RC 4919, (July 1964) IBM T. J. Watson Research Center, Yorktown Heights, New York. Also to appear in IBM Journal of Research and Development, May, 1975.

[5] D. R. Cox, "A Use of Complex Probabilities in the Theory of Stochastic Processes", Proc. Camb. Phil. Soc., Vol. 51, 1955, pp. 313-319.

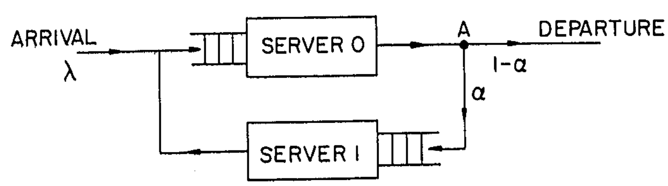

Figure 1. A cyclic queueing system.

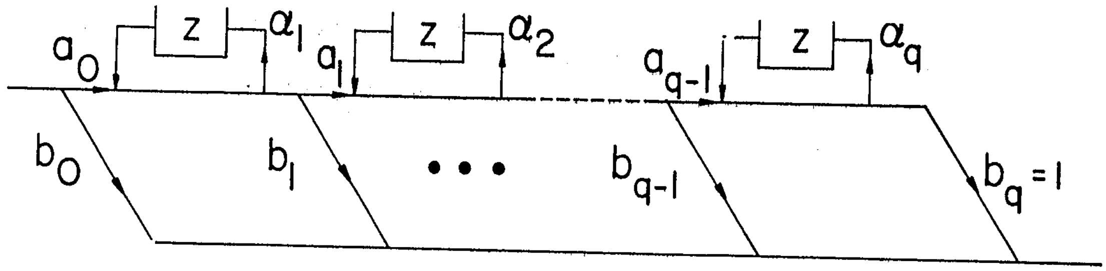

Figure 2. Schematic representation of the p.g.f. $P(z) = R(z)/Q(z)$ of the cycle distribution. Note: each box labelled z corresponds to one cycle in Figure 1.

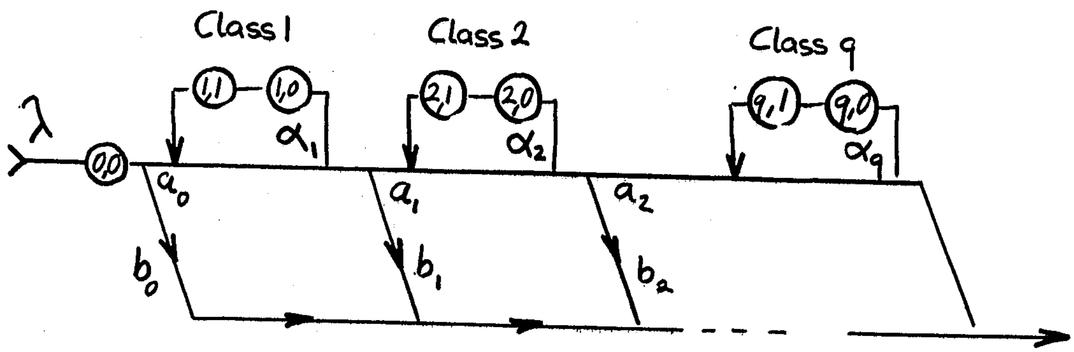

Figure 3. Equivalent multiclass queueing network for the queueing system of Figure 1 with general cycle distribution.

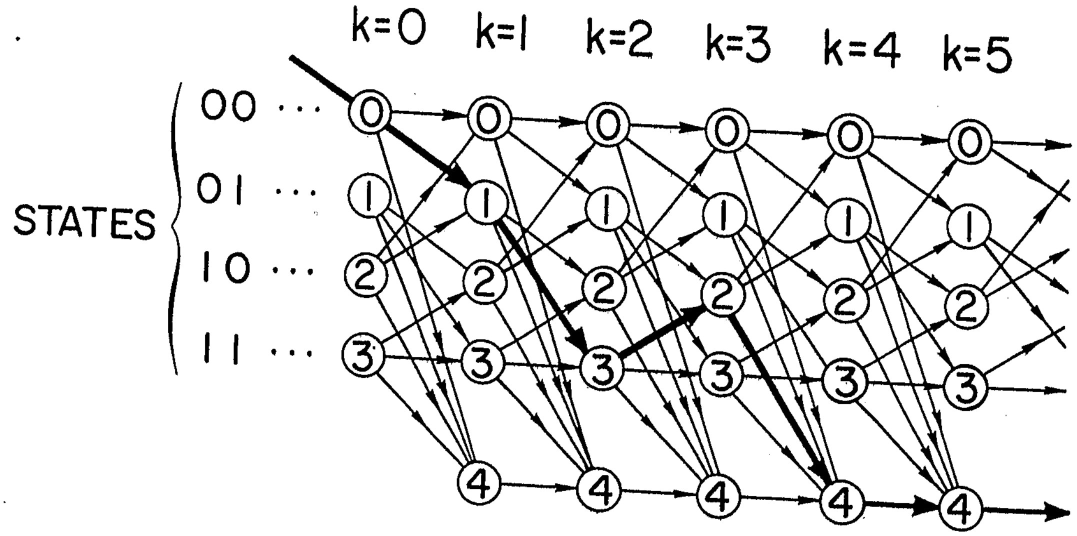

Figure 4. Trellis picture of job routing in the queueing system of Figure 1 with 2nd order Markovian routing.