SOME RESULTS ON QUEUEING NETWORK MODELS
WITH DIFFERENT CLASSES OF CUSTOMERS

by

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ABSTRACT: This paper discusses some extension of recent results obtained by Baskett, Chandy, Muntz, and Palacios on queueing network models with different classes of customers. We consider the case where the underlying Markov chain M of customer routing is decomposable into irreducible subchains. We will show that derivation of the queue-size distribution in this model is more involved than in the irreducible Markov chain model. We also treat the case where the service distributions at FCFS (first-come, first-served) centers are different for different classes of customers.

I. Introduction

Queueing network representation of multiprogrammed/ multiprocessor computer systems have been drawing[1] an increasing attention in the past few years. The original work of queueing network with exponential servers dates back to Jackson[2] and Gordon and Newell[3], but its applications to computer modeling are relatively recent (see Buzen[4] and Arora and Gallo[5]).

A noteworthy progress on this subject has been lately reported by Baskett, Muntz, Chandy and Palacios.[6,7] They have shown that the product form solution of the equilibrium queue-size distribution of Jackson is extendable to some classes of queueing networks with different classes of customers. Furthermore, they have demonstrated that the assumption of exponential service time distribution can be removed, if the corresponding service center is either an infinite-server queue, processor-sharing queue, or preemptive-resume LCFS (last-come, first-served) queue. Their product form solution of joint and marginal queue-size distribution, however, is critically dependent on an implicit assumption that the underlying Markov chain of customer transition is irreducible*.

The extension we shall make is of special interest not only as a theoretical generalization of the previous work[6,7] but for its practical value to deal with different job mixes in modeling of computer systems.

II. Statement of the Problem

Consider a queueing network system consisting of N service centers which we label as center i, $1 \le i \le N$. There exist R different classes of customers and their transitions from one center to another are governed by a first-order Markov chain M: the transition matrix is NRxNR and its element $p_{(ir)(i'r')}$ is the probability of transition from state (i,r) to state (i',r'), namely, the probability that a customer of class r which completes service at center i will next go to service center i' and changes to class r'. This underlying Markov chain M is in general decomposable into L subchains $M_1, M_2, \ldots M_L$ which are all irreducible. We can assume that $R \ge L \ge 1$ without loss of generality.

* In [7] some consideration is given to the case of decomposable Markov chain. Their solution, however, is valid only for open systems with constant arrivals.

The subchains M_ℓ's are driven by L independent Poisson arrival streams with variable rate $\lambda_\ell(m_\ell)$, where m_ℓ is the total number of customers in subchain M_ℓ at a given system state. A newly arriving customer out of stream ℓ will first enter station i with class identification r with probability $p_{\ell,(ir)}$. Similarly, a customer of class r' completing service at center i' departs the network with probability $p_{(i'r'),\ell}$. Note that our model includes the case where the network is closed with respect to subchain M_ℓ. This situation is realized by choosing $p_{\ell,(ir)} = p_{(ir)\ell} = 0$ for all $(ir) \in M_\ell$: the arrival rate $\lambda_\ell(m_\ell)$ is left undefined*.

III. Solution

3.1 Equilibrium State Distribution

Let us start with the case where all service centers are FCFS (first-come, first-served) queues and the service time distributions are exponential. We do not require that the service completion rates M_{ir}'s be common for all classes. We define the state of the system by vector $\vec{S} = [S_1, S_2, \ldots, S_i, \ldots, S_N]$, where S_i itself is a vector which represents the FCFS stack at center i: $S_i = [r_i(1), r_i(2), \ldots, r_i(j), \ldots, r_i(n_i)]$, in which $r_i(j)$ is the class of j^{th} customer at service center i, and n_i is the total number of customers at this center. Let the last entry of the FCFS stack of center i be class r, i.e., $r = r_i(n_i)$ and state (ir) belong to subchain M_ℓ.

Let $P\{\vec{S}\}$ be the equilibrium probability of state $\vec{S}$; a variable m_ℓ be the number of customers in subchain M_ℓ, and let $\vec{S}([ir]^-)$ be a state which is the same as $\vec{S}$ except that the last (i.e., n_i^{th}) entry of the stack S_i is missing. Thus a transition from state $\vec{S}([ir]^-)$ to state $\vec{S}$ is achieved when a new customer joins center i with class membership $r = r_i(n_i)$.

We define parameters e_{ir} as the average number of visits that a customer makes to service center i from its arrival at the system until its departure. It is not difficult to show[2,6,7] that e_{ir}'s are the solutions to the following set of linear equations defined for subchain M_ℓ:

$$e_{ir} = p_{\ell,(ir)} + \sum_{(i'r') \in M_\ell} e_{i'r'} p_{(i'r')(ir)}; \qquad (1)$$

* Later we formally set $\lambda_\ell(m) = 1$ for a closed subchain M_ℓ.

where $\ell=1,2,\ldots,L$. The solution of (1) is uniquely determined when $p_{\ell,(ir)}>0$ for some $(ir)\epsilon M_\ell$, i.e., when the system is open with respect to subchain M_ℓ, and M_ℓ and its absorbing state form an absorbing Markov chain. If the system is closed with respect to M_ℓ, i.e., $p_{\ell,(ir)}=0$ for all $(ir)\epsilon M_\ell$, then subchain M_ℓ is ergodic, and the solution e_{ir} is clearly nonunique. If we introduce additional condition, say,

$$\sum_{(ir)\epsilon M_\ell} e_{ir} = 1 \tag{2}$$

then the solution is uniquely given.

Then by solving "individual" balance[8] or "local" balance[9] conditions derivable from the "overall" balance equation defined for $P\{\vec{S}\}$, we obtain the following recurrence equation:

$$P\{\vec{S}\} = \lambda_\ell(m_\ell-1)\,\frac{e_{ir}}{\mu_{ir}(n_{ir})}\;P\{\vec{S}([ir]^-)\} \tag{3}$$

if the system is open with respect to M_ℓ and μ_{ir} (the service completion rate at center i given to a job in class r) is dependent on n_{ir}, the number of customers of the same class queueing at the same center; and

$$P\{\vec{S}\} = \lambda_\ell(m_\ell-1)\,\frac{e_{ir}}{\mu_{ir}(n_i)}\;P\{\vec{S}([ir]^-)\} \tag{4}$$

if the system is open with respect to M_ℓ and μ_{ir} is dependent on n_i, the total number of customers at the same center as the one being served. If the system is closed with respect to M_ℓ, relations (3) and (4) should be replaced by

$$P\{\vec{S}\} = \frac{e_{ir}}{\mu_{ir}(n_{ir})}\;P\{\vec{S}([ir]^-)\} \tag{3'}$$

and

$$P\{\vec{S}\} = \pi_\ell\frac{e_{ir}}{\mu_{ir}(n_i)}\;P\{\vec{S}([ir]^-)\} \tag{4'}$$

respectively, where constant π_ℓ is to be chosen so that $\pi_\ell e_{ir}$ is an effective arrival rate of r class jobs to center i.

By applying recurrence relation (3) or (3') repeatedly to all the entries of the FCFS stacks we obtain the equilibrium probability distribution of state $\vec{S}$ as

$$P\{\vec{S}\} = C\left(\prod_{\ell=1}^{L}\Lambda_\ell(m_\ell)\right)\prod_{\ell=1}^{L}\;\prod_{(ir)\epsilon M_\ell} f_{ir}(n_{ir}) \tag{5}$$

where

$$f_{ir}(n_{ir}) = \frac{e_{ir}^{\,n_{ir}}}{\prod_{j=1}^{n_{ir}}\mu_{ir}(j)} \tag{6}$$

and

$$\Lambda_\ell(m_\ell) = \prod_{i=0}^{m_\ell-1}\lambda_\ell(i) \tag{7}$$

If subchain M_ℓ is closed $\Lambda_\ell(m_\ell)$ is set to unity. It will be instructive to give an alternative expression of $P\{\vec{S}\}$ as

$$P\{\vec{S}\} = C\left(\prod_{\ell=1}^{L}\Lambda_\ell(m_\ell)\right)\prod_{i=1}^{N} f_i(\underline{n}_i) \tag{8}$$

where

$$\underline{n}_i = [n_{i1},\ldots,n_{ir},\ldots,n_{iR}] \tag{9}$$

and

$$f_i(\underline{n}_i) = \prod_{r=1}^{R} f_{ir}(n_{ir}) \tag{10}$$

If the service completion rate μ_{ir} is common for all classes and is dependent on n_i instead of n_{ir}, then (4) or (4') should be used and we obtain

$$f_i(\underline{n}_i) = \frac{1}{\prod_{j=1}^{n_i}\mu_i(j)}\prod_{r=1}^{R} e_{ir}^{\,n_{ir}} \tag{11}$$

The last expression was obtained earlier by Baskett and Muntz[6].

3.2 Infinite-Server and Processor-Sharing Queues with General Distributions

If service center i is an *infinite-server* queue, then the product form solution holds a general service time distribution[6,7]. The variable n_{ir} is Poisson distributed, thus the distribution is proportional to

$$f_{ir}(n_{ir}) = \frac{1}{n_{ir}!}\left(\frac{e_{ir}}{\mu_{ir}}\right)^{n_{ir}}\;;\; 1\le i\le N,\; 1\le r\le R \tag{12}$$

where $\frac{1}{\mu_{ir}}$ is the mean service time of class r customers. Note that (12) corresponds to the case where we set $\mu_{ir}(j)$ of (6) as $\mu_{ir}(j) = j\cdot\mu_{ir}$.

The joint distribution of vector $\underline{n}_i$ defined by (9) is given by a *multiple Poisson distribution*

$$f_i(\underline{n}_i) = \prod_{r=1}^{R}\frac{1}{n_{ir}!}\left(\frac{e_{ir}}{\mu_{ir}}\right)^{n_{ir}} \tag{13}$$

If service center i is *processor-sharing* (PS) the product form solution also holds for a general service time distribution[6,7,9]. A PS-queue is essentially equivalent to an infinite-server queue, except that the service completion rate is slowed down by a factor of n_i. Thus we now have

$$f_i(\underline{n}_i) = n_i!\prod_{r=1}^{R}\frac{1}{n_{ir}!}\left(\frac{e_{ir}}{\mu_{ir}}\right)^{n_{ir}} \tag{14}$$

where $\frac{1}{\mu_{ir}}$ is the mean service time of class r customer. Note that the last expression is proportional to a multinomial distribution.

Chandy[9] and Baskett and Muntz[6] have shown that preemptive-resume LCFS (last-come, first-served) queue also possesses the steady state queue sized distribution of (14).

3.3 Normalization Constants and Marginal Distributions

Evaluations of normalization constants, marginal distributions or related quantities are not so straightforward as found[6,7] for the case of irreducible Markov chain M. We start with the simplest case.

Open Network with Constant Arrivals: If the system is open with respect to all subchains M_ℓ, $\ell=1,2,\ldots,L$, and all service centers are FCFS type further simplification is possible. By denoting the constant arrival rate by λ_ℓ we have

$$\Lambda_\ell(m_\ell) = \prod_{(ir)\in M_\ell} \Lambda_\ell^{n_{ir}} \tag{15}$$

From (8) and (15) the joint distribution of the vector $\vec{n} = [\underline{n}_1, \underline{n}_2, \ldots, \underline{n}_N]$ can be obtained as a marginal distribution of $P\{\vec{S}\}$ and be written in to form

$$P\{\vec{n}\} = C \prod_{i=1}^{N} g_i(\underline{n}_i) \tag{16}$$

where

$$g_i(\underline{n}_i) = n_i! \prod_{r=1}^{R} \frac{1}{n_{ir}!} \frac{(\lambda_\ell e_{ir})^{n_{ir}}}{\prod_{j=1}^{n_{ir}} \mu_{ir}(j)} \tag{17}$$

and $n_i = \sum_{r=1}^{R} n_{ir}$. The subscript ℓ of λ_ℓ is such that $(ir)\in M_\ell$. The derivation of (17) is based on the fact that any two distinct states of the FCFS stack at center i, that are equivalent except for permutation of their dements, have the same probability. If in particular $\mu_{ir}(j)=\mu_{ir}$, then (17) reduces a multinomial distribution:

$$g_i(\underline{n}_i) = n_i! \prod_{r=1}^{R} \frac{1}{n_{ir}!} \left(\frac{\lambda_\ell e_{ir}}{\mu_{ir}}\right)^{n_{ir}} \tag{18}$$

if service center is a FCFS queue with common exponential service time distributions, where service completion rate μ_{ir} can be dependent on n_i, function $g_i(\underline{n}_i)$ is obtained from (11) as follows[7]

$$g_i(\underline{n}_i) = \frac{n_i!}{\prod_{j=1}^{n_i} \mu_i(j)} \prod_{r=1}^{R} \frac{1}{n_{ir}!} (\lambda_\ell e_{ir})^{n_{ir}} \tag{19}$$

In a similar manner it can be shown from the result of Section 3.2 that if service center i is a processor sharing queue or a preemptive-resume LCFS queue $g_i(\underline{n}_i)$ is given also by (18). Note, however, that this solution holds even for any general service time distribution with means $1/\mu_{ir}$. Similarly if service center i is an infinite-server queue we have

$$g_i(\underline{n}_i) = \prod_{i=1}^{R} \frac{1}{n_{ir}!} \left(\frac{\lambda_\ell e_{ir}}{\mu_{ir}}\right)^{n_{ir}} \tag{20}$$

which holds for any general service-time distribution with mean $1/\mu_{ir}$.

The joint generating function of (16) is given by

$$Q(z) = E\left[\prod_{i=1}^{N}\prod_{r=1}^{R} z_{ir}^{n_{ir}}\right]$$

$$= C \prod_{i=1}^{N} G_i(\underline{z}_i) \tag{21}$$

where

$$G_i(\underline{z}_i) = \sum_{\underline{n}_i} \prod_{r=1}^{R} z_{ir}^{n_{ir}} g_i(\underline{n}_i) \tag{22}$$

By inserting (18) into (22) we readily obtain

$$G_i(\underline{z}_i) = \sum_{n_i=0}^{\infty} \left(\sum_{r=1}^{R} \frac{\lambda_\ell e_{ir} z_{ir}}{\mu_{ir}}\right)^{n_i} = \left\{1 - \sum_{r=1}^{R} \frac{\lambda_\ell e_{ir} z_{ir}}{\mu_{ir}}\right\}^{-1} \tag{23}$$

for FCFS, PS and LCFS. Likewise from (20) and (22)

$$G_i(\underline{z}_i) = \prod_{r=1}^{R} \sum_{n_i=0}^{\infty} \frac{1}{n_{ir}!} \left(\frac{\lambda_\ell e_{ir} z_{ir}}{\mu_{ir}}\right)^{n_{ir}}$$

$$= \exp\left\{\sum_{r=1}^{R} \frac{\lambda_\ell e_{ir} z_{ir}}{\mu_{ir}}\right\} \tag{24}$$

for an infinite-server center.

For a queue dependent FCFS center (but with a common distribution function for different classes), we define a function

$$\Phi_i(z) = \sum_{n=0}^{\infty} \frac{z^n}{\prod_{j=1}^{n_i} \mu_i(j)} \tag{25}$$

Then from (22) and (25) we have

$$G_i(\underline{z}_i) = \Phi_i\left(\sum_{r=1}^{R} \lambda_\ell e_{ir} z_{ir}\right) \tag{26}$$

The *probability generating function* (p.g.f.) of $\underline{n}_i=[n_{i1},\ldots,n_{iR}]$ is proportional to $G_i(\underline{z}_i)$

defined above. By its definition, a p.g.f. must be unity when their arguments are all unity. Hence the p.g.f. of $\underline{n}_i$ is given by $G_i(\underline{z}_i)/G_i(\underline{1})$, where $\underline{1}$ is an R-dimensional vector whose entries are all 1.

The p.g.f. $H_i(z_i)$ of $n_i = \sum_{r=1}^{R} n_{ir}$ is obtained by setting $z_{ir}=z_i$ in the p.g.f. of $\underline{n}_i$. Thus from (23) we obtain

$$H_i(z_i) = \frac{1-\sum_{r=1}^{R} \frac{\lambda_\ell e_{ir}}{\mu_{ir}}}{1-z_i \sum_{r=1}^{R} \frac{\lambda_\ell e_{ir}}{\mu_{ir}}} \tag{27}$$

for FCFS, PS and FCFS, which yields the following geometric distribution

$$h_i(n_i) = \left(1-\sum_{r=1}^{R} \frac{\lambda_\ell e_{ir}}{\mu_{ir}}\right)\left(\sum_{r=1}^{R} \frac{\lambda_\ell e_{ir}}{\mu_{ir}}\right)^{n_i} \tag{28}$$

For an infinite-server center we obtain from (24)

$$H_i(z_i) = \exp\left\{(z_i-1)\sum_{r=1}^{R} \frac{\lambda_\ell e_{ir}}{\mu_{ir}}\right\} \tag{29}$$

which leads to the following Poisson distribution

$$h_i(n_i) = \frac{1}{n_i!}\left(\sum_{r=1}^{R} \frac{\lambda_\ell e_{ir}}{\mu_{ir}}\right)^{n_i} \exp\left\{-\sum_{r=1}^{R} \frac{\lambda_\ell e_{ir}}{\mu_{ir}}\right\} \tag{30}$$

Similarly for a queue dependent FCFS center we use (26):

$$H_i(z_i) = \Phi_i\left(z_i \sum_{r=1}^{R} \lambda_\ell e_{ir}\right) \Big/ \Phi_i\left(\sum_{r=1}^{R} \lambda_\ell e_{ir}\right) \tag{31}$$

which yields, by inversion,

$$h_i(n_i) = \left(\sum_{r=1}^{R} \lambda_\ell e_{ir}\right)^{n_i} \Big/ \Phi_i\left(\sum_{r=1}^{R} \lambda_\ell e_{ir}\right) \prod_{j=1}^{n_i} \mu_i(j) \tag{32}$$

It will be instructive to note that $h_i(n_i)$ of (28) and (30) correspond to the queue size distributing of an M/M/1 and M/G/∞, respectively, with traffic intensity $\rho_i = \sum_{r=1}^{R} \frac{\lambda_\ell e_{ir}}{\mu_{ir}}$.

By now it is obvious that the constant C of (21) is obtainable as

$$C^{-1} = \prod_{i=1}^{N} G_i(\underline{1}) \tag{33}$$

If we wish to compute the marginal distribution of the number of customers in subchain M_ℓ, we set in (21) $z_{ir}=z_\ell$ if $(ir)\in M_\ell$ and $z_{ir}=1$, otherwise by inverting the resultant function, we have the desired distribution. Similarly, the distribution of the total number of customers in the system is obtained by setting all $z_{ir}=z$ in (21).

Closed Network:

Now we want to obtain the queue-size distribution of a closed network. Let the network be closed with respect to all subchains M_ℓ, $\ell=1,2,\dots,L$. The number of customers in each chain is fixed which we denote by m_ℓ, $\ell=1,2,\dots,L$. Therefore, the total number of customers in the system $n = \sum_{\ell=1}^{L} m_\ell = \sum_{i=1}^{N}\sum_{r=1}^{R} n_{ir}$ is also fixed. Let F be a set of vectors $\vec{n}$ which are feasible in this closed network:

$$F = \left\{ \vec{n}: \sum_{(ir)\in M_\ell} n_{ir} = m_\ell \ ;\ell=1,2,\dots,L \right\} \tag{34}$$

Over this feasible set F is defined the recurrence relation (3') or (4'). Equations (3') and (4') can be viewed as equivalent to (3) and (4), respectively, in which $\lambda_\ell(.)=\pi_\ell$. How to determine the constant parameters π_ℓ will be demonstrated shortly. Then by taking the same steps that have led us to (16), we obtain the following joint queue size distribution for a closed network

$$P^*\{\vec{n}\} = \begin{cases} C^* \prod_{i=1}^{N} g_i^*(\underline{n}_i) & : \underline{n}\in F \\ 0 & : \underline{n}\notin F \end{cases} \tag{35}$$

where $g_i^*(\underline{n}_i)$ is the same as that of (17) - (20) except that the λ_ℓ are now replaced by π_ℓ. The joint generating function $Q^*(\vec{z})$ consists of those terms $\prod_{i=1}^{N}\prod_{r=1}^{R} z_{ir}^{n_{ir}}$ in which $\sum_{(ir)\in M_\ell} n_{ir} = m_\ell$. In order to derive this from $Q(\vec{z})$ of (21), we replace z_{ir} of (21) by $\theta_\ell z_{ir}$ for all $(ir)\in M_\ell$ $\ell=1,2,\dots,L$ and then differentiate it m_ℓ times with respect to θ_ℓ, and set $\theta_\ell=0,\ell=1,2,\dots,L$.

$$Q^*(\vec{z}) = \frac{C^*}{\prod_{\ell=1}^{L} m_\ell!}\left[\left(\partial^n/\partial\theta_1^{m_1}\dots\partial\theta_L^{m_L}\right) Q\left(\{\theta_\ell z_{ir}\}\right)\right]_{\theta_1=\dots=\theta_L=0} \tag{36}$$

Example Consider a simple closed network consisting of three service centers. There are two classes of customers which we denote by a and b: class a cycles around centers 1 and 2, and class b cycles around centers 2 and 3. Thus, we have two subchains and the set of states (i,r)'s is decomposable into

$M_a = \{(1,a), (2,a)\}$

$M_b = \{(1,b), (3,b)\}$

We assume that center 2 is a PS queue with mean service time $\frac{1}{\mu_{ir}}$, r=a,b ; center 2 is a FCFS queue with service completion rate μ_{2a} and center 3 is an infinite-server queue with mean service time $\frac{1}{\mu_{3b}}$. Then from (23) and (24) and using $e_{ir}=1/2$ for all (ir), we obtain

$$G_1(z_{1a},z_{1b}) = \left(1 - \frac{\pi_a z_{1a}}{2\mu_{1a}} - \frac{\pi_i z_{1b}}{2\mu_{1b}}\right)^{-1}$$

$$G_2(z_{2a}) = \left(1 - \frac{\pi_a z_{2a}}{2\mu_{2a}}\right)^{-1}$$

and

$$G_3(z_{3b}) = \exp\{\frac{\pi_b z_{3b}}{2\mu_{3b}}\}$$

Then by substituting these results into (36) we have

$$Q^*(z) = \frac{C^*}{m_a!\,m_b!}\left[\left(\partial^{m_a+m_b}\Big/\partial\theta_a^{m_a}\partial\theta_b^{m_b}\right)\cdot \left\{\left(1 - \frac{\pi_a\theta_a z_{1a}}{2\mu_{1a}} - \frac{\pi_b\theta_b z_{1b}}{2\mu_{1b}}\right)^{-1}\left(1 - \frac{\pi_a\theta_a z_{2a}}{2\mu_{2a}}\right)^{-1} \exp\left(\frac{\pi_b\theta_b z_{3b}}{2\mu_{3b}}\right)\right\}\right]_{\theta_a=\theta_b=0}$$

The unknown constants C^*, π_a and π_b are obtained from the following conditions: (1) The probabilities sum up to one, i.e., $Q^*(\underline{1})=1$; (2) The inflow rate of each class must equal its outflow rate at each service center. We will not pursue this procedure here because of space limitation.

IV Concluding Remarks

Throughout the present paper, we resorted primarily to the generating function method. For actual numerical evaluating of the solution, however, the transformation domain is not necessarily best in terms of computational efficiency. The recursive algorithm, recently reported by Buzen[10] and independently by the present authors[11,12] are directly extendable to the open system discussed in the present paper. For the closed network of the type discussed in the last section, however, a further study is required for computation efficiency.

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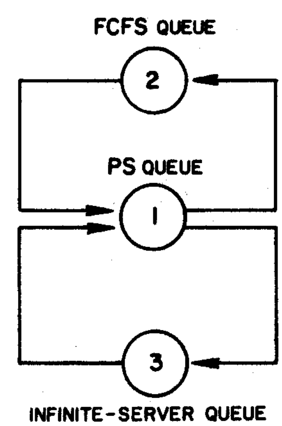

Figure 1. A closed network with 3 service centers, 2 classes of customers (class a and class b).