APPLICATION OF HESTENES-STIEFEL ALGORITHM
TO CHANNEL EQUALIZATION

by

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Summary

The conjugate gradient algorithm of Hestenes and Stiefel is applied to the design of a time-domain equalizer in a form which requires no prior knowledge of statistics such as correlation function or power spectrum. A practical equalizer structure is discussed and a bound for the rate of convergence is obtained using Kantorovich inequality.

Introduction

Consider a PAM baseband system given in Figure 1. The equalizer $\underline{w} = \{w(u);\ -L_1 \le u \le L_2\}$ is a linear discrete filter with L stages, where $L = L_1 + L_2 + 1$, and its input and output are related by

$$y(k) = \sum_{u=-L_1}^{L_2} w(u)x(k-u) \quad (1)$$

We define the squared error distortion criterion[1,2] by

$$P = \sum_k \{y(k) - s(k)\}^2 = 2J + \sum_k s^2(k) \quad (2)$$

where J is the following quadratic function of L-dimensional variable $\underline{w}$.

$$J = \frac{1}{2} \langle \underline{w}, \underline{R}\,\underline{w} \rangle - \langle \underline{w}, \underline{b} \rangle \quad (3)$$

Here $\langle\ ,\ \rangle$ means inner-product. $\underline{R}$ is an $L \times L$ matrix whose entries are given by

$$R(u,v) = \sum_k x(k-u)x(k-v), \quad -L_1 \le u,v \le L_2 \quad (4)$$

and $\underline{b}$ is an $L \times 1$ matrix:

$$b(u) = \sum_k x(k-u)s(k), \quad -L_1 \le u \le L_2 \quad (5)$$

It is clear that $\underline{R}$ is a non-negative definite symmetric matrix and the value $\underline{w}^*$ that minimizes J is unique if $\underline{R}$ is positive definite.

Hestenes-Stiefel Algorithm

The conjugate gradient method due to Hestenes and Stiefel[3] is an iterative algorithm to solve a set of simultaneous equations

$$\underline{R}\,\underline{w} = \underline{b} \quad (6)$$

where $\underline{R}$ is an $L \times L$ positive definite matrix. This method is an L-step iterative one, i.e., the algorithm is applied to give successive approximations to the solution and, if computations are done with complete accuracy, the solution is obtained after M iterations where $M \le L$. A full and lucid description of Hestenes-Stiefel algorithm has been given by Beckman[4].

Clearly the same algorithm can be applied to find $\underline{w}$ that minimizes P of (2). Let $\underline{w}_0$ be an arbitrary starting approximation to the solution vector of (6). The direction of the first move, $\underline{p}_0$, is the same as in the steepest descent method, i.e.,

$$\underline{p}_0 = \underline{r}_0 = \underline{b} - \underline{R}\,\underline{w}_0 \quad (7)$$

Then the following iterative formulae (for $i \ge 0$) give successive approximations $\{\underline{w}_i\}$ that lead to the solution $\underline{w}^* = \underline{R}^{-1}\underline{b}$:

$$\alpha_i = |\underline{r}_i|^2 / \langle \underline{p}_i, \underline{R}\,\underline{p}_i \rangle \quad (8a)$$

$$\underline{w}_{i+1} = \underline{w}_i + \alpha_i\,\underline{p}_i \quad (8b)$$

$$\underline{r}_{i+1} = \underline{b} - \underline{R}\,\underline{w}_{i+1} = \underline{r}_i - \alpha_i\,\underline{R}\,\underline{p}_i \quad (8c)$$

$$\beta_i = |\underline{r}_{i+1}|^2 / |\underline{r}_i|^2 \quad (8d)$$

$$\underline{p}_{i+1} = \underline{r}_{i+1} + \beta_i\,\underline{p}_i \quad (8e)$$

In the above formulae $\underline{r}_i$ is the gradient of J at point $\underline{w}_i$, $|\underline{r}_i|$ is its Euclidean norm, $\underline{p}_i$ is the direction of search to the next approximation, and α_i is the optimum step size.

After M iterations with $M \le L$, $\underline{w}_M$ will be equal to the solution $\underline{w}^*$ if all computations are done with no loss of accuracy. Many relations hold among the quantities appearing in (8a) through (8e):

$$\langle \underline{r}_i, \underline{r}_j \rangle = 0 \qquad i \ne j \quad (9a)$$

$$\langle \underline{p}_i, \underline{R}\,\underline{p}_j \rangle = 0 \qquad i \ne j \quad (9b)$$

$$\langle \underline{p}_i, \underline{r}_j \rangle \begin{cases} = 0 & i < j \quad (9c) \\ = |\underline{r}_i|^2 & i \ge j \quad (9d) \end{cases}$$

$$\langle \underline{r}_i, \underline{R}\,\underline{p}_i \rangle = \langle \underline{p}_i, \underline{R}\,\underline{p}_i \rangle \quad (9e)$$

$$\langle \underline{r}_i, \underline{R}\,\underline{p}_j \rangle = 0 \quad (9f)$$

Using the properties (9a), (8c), and (9d), we obtain alternative expressions for α_i and β_i.

$$\alpha_i = \langle \underline{r}_i, \underline{p}_i \rangle / \langle \underline{p}_i, \underline{R}\, \underline{p}_i \rangle \tag{10a}$$

$$\beta_i = - \langle \underline{r}_{i+1}, \underline{R}\, \underline{p}_i \rangle / \langle \underline{p}_i, \underline{R}\, \underline{p}_i \rangle \tag{10b}$$

Equation (9b) shows that the direction vectors $\{\underline{p}_i : i = 0, 1, \ldots, L-1\}$ form a set of R-conjugate or R-orthogonal vectors and span the space E^L. The squared error distortion decreases at each step of the iteration:

$$J_{i+1} - J_i = - \frac{1}{2} \langle \underline{p}_i, \underline{r}_i \rangle^2 / \langle \underline{p}_i, \underline{R}\, \underline{p}_i \rangle = - \frac{1}{2} |\underline{r}_i|^4 / \langle \underline{p}_i, \underline{R}\, \underline{p}_i \rangle \tag{11}$$

Some other salient relations that hold among the quantities appearing the iterative formulae are

$$|\underline{w}_i - \underline{w}^*| > |\underline{w}_j - \underline{w}^*| \qquad i > j \tag{12}$$

and

$$\langle \underline{w}_i - \underline{w}^*, \underline{R}^{-1}(\underline{w}_i - \underline{w}^*) \rangle > \langle \underline{w}_j - \underline{w}^*, \underline{R}^{-1}(\underline{w}_j - \underline{w}^*) \rangle \qquad i < j \tag{13}$$

These results indicate that if we stop the iterative process at any step, the last obtained approximation is the best in the sense of being the closest to the true solution, whether the metric is defined by either $\langle \underline{x}, \underline{x} \rangle^{1/2}$ or by $\langle \underline{x}, \underline{R}^{-1} \underline{x} \rangle^{1/2}$.

Equalizer Structure

Although (7) and (8) appear to require computation of the auto-correlation function $R(u,v)$, these formulae can be written in the following form by substituting the definition (4) Initialization:

$$w_0(u) \text{ is arbitrary, } -L_1 \le u \le L_2 \tag{14a}$$

$$p_0(u) = r_0(u) = - \sum_k x(k-u) e_0(k) \tag{14b}$$

where $e_0(k)$ is the error of the initial estimate $y_0(k)$:

$$e_0(k) = y_0(k) - s(k) \tag{14c}$$

For $i \ge 0$:

$$q_i(k) = \sum_{u=-L_1}^{L_2} p_i(u) x(k-u) \tag{15a}$$

$$\alpha_i = |\underline{r}_i|^2 / ||q_i(\cdot)||^2 \tag{15b}$$

$$\underline{w}_{i+1} = \underline{w}_i + \alpha_i \underline{p}_i \tag{15c}$$

$$y_{i+1}(k) = \sum_{u=-L_1}^{L_2} w_{i+1}(u) x(k-u) \tag{15d}$$

$$e_{i+1}(k) = y_{i+1}(k) - s(k) \tag{15e}$$

$$r_{i+1}(u) = - \sum_k x(k-u) e_{i+1}(k) \tag{15f}$$

$$\beta_i = |\underline{r}_{i+1}|^2 / |\underline{r}_i|^2 \tag{15g}$$

$$\underline{p}_{i+1} = \underline{r}_{i+1} + \beta_i \underline{p}_i \tag{15h}$$

Sequence $q_i(h)$ of (15a) is obtained by passing the sequence $x(k)$ into a transversal filter $\underline{p}_i$ and $||q_i(\cdot)||^2$ is its norm square or energy.

Figure 2 is a block diagram of an automatic equalizer (i.e., $s(k)$ is known) or an adaptive equalizer (in which the decision output $\hat{s}(k)$ is used instead of $s(k)$) based on the iterative procedure (14a) through (15h). The transversal filter $w_{i+1}(u)$ and the cross-correlator are common to most of the existing equalizers[1,2]. The major additional hardware is a transversal filter $p_i(u)$. Other elements in the block diagram are mostly for the purpose of storing vectors and scalar numbers. The operation of each block will be self-explanatory by referring to the formulae (14) and (15).

If we set $\beta_i = 0$, the structure is equivalent to the steepest descent type equalizer. If we use only blocks marked by * and replace α_i by a constant, then the structure is reduced to the equalizer studied by the previous authors[1,2]. It has been shown in many optimization problems[5] and in the previous study of array processors[6,7] that the H-S algorithm provides a faster convergence than the usual gradient or steepest descent method.

Rate of Convergence

Although we know that the algorithm converges to the optimal solution in a finite number of steps, we are also interested in the convergence behavior of the first few iterations. We will obtain an expression for bound on the rate of convergence and will see that the ratio of the greatest and smallest eigenvalues of the matrix $\underline{R}$ determines the speed of convergence.

We obtain from (11) that the squared error distortion sequence P_i decreases according to the relation

$$P_{i+1} = P_i - \langle \underline{r}_i, \underline{r}_i \rangle^2 / \langle \underline{p}_i, \underline{R}\, \underline{p}_i \rangle \tag{16}$$

On rewriting P_i as

$$P_i = \langle \underline{b} - \underline{R}\, \underline{w}_i, \underline{R}^{-1}(\underline{b} - \underline{R}\, \underline{w}_i) \rangle + ||s(\cdot)||^2 - \langle \underline{b}, \underline{R}^{-1} \underline{b} \rangle \tag{17}$$

we obtain the minimum value of $\{P_i\}$ as follows:

$$P^* = ||s(\cdot)||^2 - \langle \underline{w}^*, \underline{b} \rangle = [s(\cdot) - y^*(\cdot), s(\cdot)] \tag{18}$$

where $[x(\cdot), y(\cdot)]$ is an inner-product of sequences $x(h)$ and $y(h)$. $y^*(k)$ of (18) is the output of the optimum equalizer $\underline{w}^*$. Alternative expression of P^* is obtained by using $\underline{b} = \underline{R}\, \underline{w}$:

$$P^* = ||s(\cdot)||^2 - \langle \underline{w}^*, \underline{R}\,\underline{w}^* \rangle = ||s(\cdot)||^2 - ||y^*(\cdot)||^2 \qquad (19)$$

From (8c) and (19) we obtain

$$P_i = \langle \underline{r}_i, \underline{R}^{-1}\underline{r}_i \rangle + P^* \qquad (20)$$

Therefore by substituting (20) into (11) we have

$$\begin{aligned} P_{i+1} - P^* &= P_i - P^* - |\underline{r}_i|^4 / \langle \underline{p}_i, \underline{R}\,\underline{p}_i \rangle \\ &= (P_i - P^*)\{1 - |\underline{r}_i|^4 / \langle \underline{p}_i, \underline{R}\,\underline{p}_i \rangle \langle \underline{r}_i, \underline{R}^{-1}\underline{r}_i \rangle\} \end{aligned} \qquad (21)$$

By using a simple manipulation, we have

$$\frac{|\underline{r}_i|^2}{\langle \underline{r}_i, \underline{R}\,\underline{r}_i \rangle} = \frac{|\underline{r}_i|^2}{\langle \underline{p}_i, \underline{R}\,\underline{p}_i \rangle} \cdot \frac{\langle \underline{p}_i, \underline{R}\,\underline{p}_i \rangle}{\langle \underline{r}_i, \underline{R}\,\underline{r}_i \rangle} = f_i \frac{|\underline{r}_i|^2}{\langle \underline{p}_i, \underline{R}\,\underline{p}_i \rangle} \qquad (22)$$

Then from (21) and (22)

$$P_{i+1} - P^* = (P_i - P^*)\{1 - \frac{|\underline{r}_i|^4}{\langle \underline{r}_i, \underline{R}\,\underline{r}_i \rangle \langle \underline{r}_i, \underline{R}^{-1}\underline{r}_i \rangle} f_i\} \qquad (23)$$

where f_i in (22) and (23) is

$$f_i = \frac{\langle \underline{r}_i, \underline{R}\,\underline{r}_i \rangle}{\langle \underline{p}_i, \underline{R}\,\underline{p}_i \rangle} \qquad (24)$$

It can be shown from (8e) and (9b) that $f_i > 1$.

Let the largest and smallest eigenvalues of the $L \times L$ matrix $\underline{R}$ be μ_1 and μ_L, respectively. Then applying Kantorovich inequality (Appendix A)

$$|\underline{r}|^4 \le \langle \underline{r}, \underline{R}\,\underline{r} \rangle \langle \underline{r}, \underline{R}^{-1}\underline{r} \rangle \le \frac{(\mu_1 + \mu_L)^2}{4\,\mu_1\,\mu_L} |\underline{r}|^4 \qquad (25)$$

to (23), we obtain an upper bound on P_i:

$$P_i < P^* + \rho^{2i}(P_0 - P^*) \qquad (26)$$

where

$$\rho = \left(\frac{\mu_1 - \mu_L}{\mu_1 + \mu_L} \right) \qquad (27)$$

A spectral density $P(\lambda)$ can be defined for the auto-correlation function $R(u)$ of a discrete process $x(k)$.

$$R(u) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{+iu\lambda} P(\lambda) d\lambda, \quad u = 0, \pm 1, \pm 2, \ldots \qquad (28)$$

A quadratic form in the covariance matrix $\underline{R}$ can then be expressed in the spectrum form

$$\langle \underline{x}, \underline{R}\,\underline{x} \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \sum_{u=-L_1}^{L_2} x(u) e^{+iu\lambda} \right|^2 P(\lambda) d\lambda \qquad (29)$$

Using the relations

$$\inf_{\lambda} P(\lambda) \le P(\lambda) \le \sup_{\lambda} P(\lambda) \qquad (30)$$

and

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \sum_{u=-L_1}^{L_2} x(u) e^{+iu\lambda} \right|^2 d\lambda = \sum_{u=-L_1}^{L_2} |x(u)|^2 = |x|^2 \qquad (31)$$

We obtain the following expression which holds independent of L_1 and L_2.

$$\inf_{\lambda} P(\lambda) \le \frac{\langle \underline{x}, \underline{R}\,\underline{x} \rangle}{|\underline{x}|^2} \le \sup_{\lambda} P(\lambda) \qquad (32)$$

Therefore the maximum and minimum eigenvalues of $\underline{R}$ are bounded as follows:

$$\mu_L \ge \inf_{\lambda} P(\lambda) \qquad (33)$$

$$\mu_1 \le \sup_{\lambda} P(\lambda) \qquad (34)$$

In fact, as L_1 and L_2 go to infinity, the maximum and minimum eigenvalues attain the bounds[8]. From (27) and (33) we obtain

$$\rho \le \frac{\sup P(\lambda) - \inf P(\lambda)}{\sup P(\lambda) + \inf P(\lambda)} \qquad (35)$$

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Appendix A

Kantorovich Inequality[9,10] of Equation (25)

We assume without loss of generality that $|\underline{v}| = 1$. Then we want to prove the following inequality.

$$1 \leq \langle \underline{v}, \underline{R}\, \underline{v} \rangle \langle \underline{v}, \underline{R}^{-1} \underline{v} \rangle \leq \frac{(\mu_1 + \mu_L)^2}{4\, \mu_1\, \mu_L} \qquad \text{(A-1)}$$

with

$$|\underline{v}| = 1 \qquad \text{(A-2)}$$

where

$$\mu_1 \geq \mu_2 \geq \ldots \geq \mu_L > 0 \qquad \text{(A-3)}$$

are the eigenvalues of a positive definite matrix $\underline{R}$. Let $\underline{u}_1, \underline{u}_2, \ldots, \underline{u}_L$ be orthonormal eigenvectors corresponding to eigenvalues of (A-3).

Consider the (x-y) plane with L points located at (μ_i, μ_i^{-1}), $i = 1, 2, \ldots, L$. They are on the curve $x\, y = 1$ and these L points span a convex region S. Let

$$x^* = \langle \underline{v}, \underline{R}\, \underline{v} \rangle = \sum_{i=1}^{L} a_i^2\, \mu_i \qquad \text{(A-4)}$$

and

$$y^* = \langle \underline{v}, \underline{R}^{-1} \underline{v} \rangle = \sum_{i=1}^{L} a_i^2\, \mu_i^{-1} \qquad \text{(A-5)}$$

where

$$a_i = \langle \underline{v}, \underline{u}_i \rangle \qquad \text{(A-6)}$$

The point (x^*, y^*) lies in the convex region S for any vector $\underline{v}$, since

$$\sum_{i=1}^{L} a_i^2 = \sum_{i=1}^{L} \langle \underline{v}, \underline{u}_i \rangle^2\; |\underline{v}|^2 = 1 \qquad \text{(A-7)}$$

Therefore the maximum value of $x^* y^*$ is given by c, where $x\, y = c$ touches the line connecting the points (μ_1, μ_1^{-1}) and (μ_L, μ_L^{-1}). The value c can be obtained as

$$c = \frac{(\mu_1 + \mu_L)^2}{4\, \mu_1\, \mu_L} \qquad \text{(A-8)}$$

It is easy to see that the minimum value of $x^* y^*$ is unity. Thus Kantorovich inequality of (A-1) has been proved.

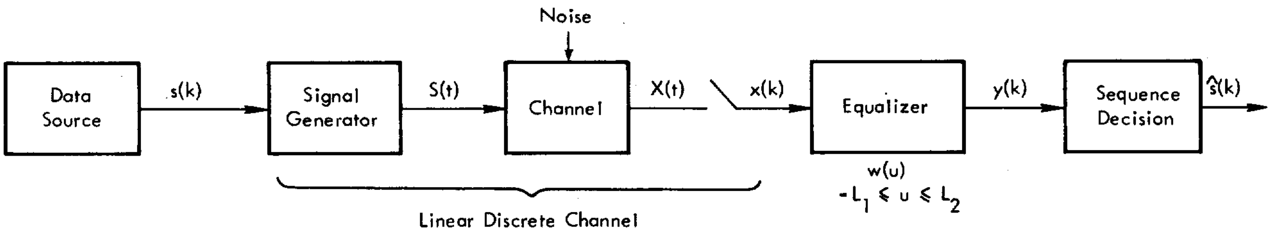

Figure 1

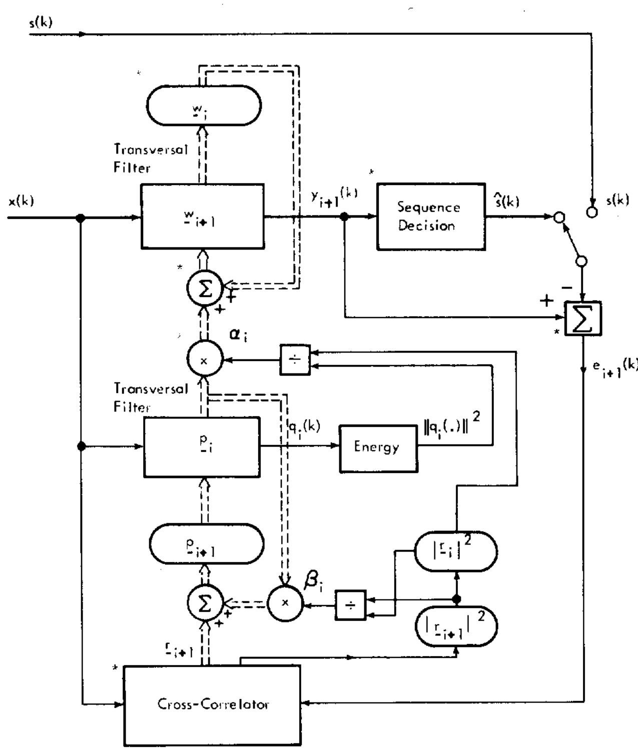

Figure 2