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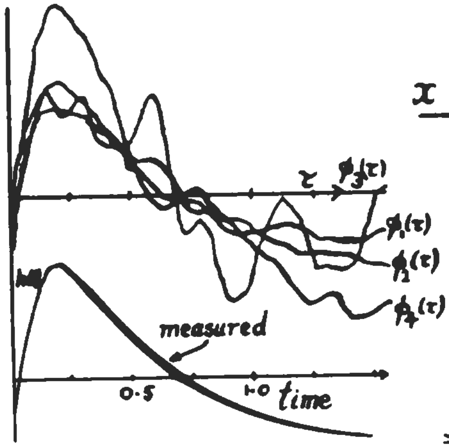

Figure 5. cross-correlation functions and the impulse response. (from reference 38)

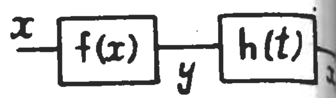

Figure 6. a particular non-linear system.

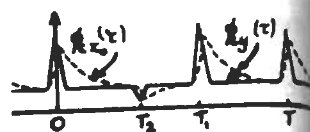

Figure 7 cross-correlation function for non-linear system. (from reference 18)

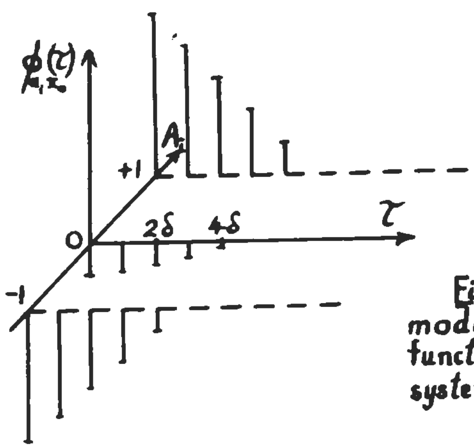

Figure 8. modified cross-correlation functions for a non-linear system

THE GENERALIZED CRAMER-RAO INEQUALITY AND ITS APPLICATION TO PARAMETER ESTIMATION

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ABSTRACT

The correct procedure for estimating unknown parameters is a conditional estimation when no a priori information concerning the possible values of the parameters is available. It is well known that the Cramer-Rao inequality gives a lower bound on the variance of conditional estimates. In many cases the observer has some a priori knowledge concerning the value of the unknown parameters. Under these circumstances the estimates should be unconditional. In this treatment the Cramer-Rao inequality is extended to the simultaneous estimation of many parameters (real or complex) when a priori knowledge is available. Lower bounds are given in terms of the averaged Fisher's information measure matrix.

The results are applied to the estimation of pulse trains. The attainable accuracy in the simultaneous estimation of signal strength, time of arrival, and doppler-shift frequency is given in terms of quantities related to the signal waveform. It is shown finally that these lower bounds are attained asymptotically by the maximum likelihood estimator as the output signal-to-noise ratio increases.

1. INTRODUCTION

The correct procedure for estimating unknown parameters is a conditional estimate [1] when no a priori information concerning the possible values of the parameters is available. It is well known that the Cramer-Rao inequality gives a lower bound on the variance of conditional estimates. Let $\hat{a}(x)$ be any conditional estimate of a single unknown parameter α (real or complex) such that

$$E_x[\hat{a}(x)] = \int_X \hat{a}(x)p(x/\alpha)dx = \alpha + b(\alpha) \qquad (1.1)$$

where $b(\alpha)$ is called the bias of the estimator $\hat{a}(x)$. Then the variance of the estimate satisfies the following inequality under some regularity conditions [2].

$$\operatorname{var}_x[\hat{a}(x)] \ge \frac{|1 + b'(\alpha)|^2}{I(\alpha)} \qquad (1.2)$$

where $I(\alpha)$ is Fisher's information measure [2,4,5] given by

$$I(\alpha) = E_x\left|\frac{\partial \log p(x/\alpha)}{\partial\alpha}\right|^2 \qquad (1.3)$$

The equality in Eq. (1.2) holds only when, for all $x\epsilon X$,

$$\hat{a}(x) - \alpha - b(\alpha) = k(\alpha)\left\{\frac{\partial \log p(x/\alpha)}{\partial\alpha}\right\}^* \qquad (1.4)$$

where $k(\alpha)$ is a function which does not depend on x, and * indicates complex conjugate.

A lower bound for the mean square error between the estimate $\hat{\alpha}(x)$ and the true value α is readily given by

$$\underset{x}{E} \, |\hat{\alpha}(x) - \alpha|^2 \geq |b(\alpha)|^2 + \frac{|1 + b'(\alpha)|^2}{I(\alpha)} \tag{1.5}$$

Particularly in the case of an unbiased estimator, Eqs. (1.2) and (1.5) reduce to

$$\underset{x}{E} \, |\hat{\alpha}(x) - \alpha|^2 = \underset{x}{\mathrm{var}} \, \hat{\alpha}(x) \geq \frac{1}{I(\alpha)} \tag{1.6}$$

2. CONDITIONAL ESTIMATE OF MULTIPARAMETERS

Assume that the probability density functions $p(x/\underline{\alpha}) = p(x/\alpha_1, \alpha_2, ---, \alpha_k)$, $x \epsilon X$, for the observation x, is known except for a finite number of unknown but fixed parameters $\alpha_1, \alpha_2, ---, \alpha_k$. Let $\hat{\alpha}_i(x)$ be a conditional estimate of α_i having a bias $b_i(\alpha)$, i.e.

$$\int \hat{\alpha}_i(x) p(x/\underline{\alpha}) dx = \alpha_i + b_i(\underline{\alpha}) \quad , \quad i = 1, ---, k \tag{2.1}$$

We assume the following regularity condition:

$$\frac{\partial}{\partial \alpha_j} \int \hat{\alpha}_i(x) p(x/\underline{\alpha}) dx = \int \hat{\alpha}_i(x) \frac{\partial}{\partial \alpha_j} p(x/\underline{\alpha}) dx, \quad i,j, = 1, \ldots, k \tag{2.2a}$$

which is equivalent to

$$\delta_{ij} + \frac{\partial b_i(\underline{\alpha})}{\partial \alpha_j} = \int \hat{\alpha}_i(x) \frac{\partial \log p(x/\underline{\alpha})}{\partial \alpha_j} p(x/\underline{\alpha}) dx \tag{2.2b}$$

Then the covariance matrix of a set of complex random variables $\hat{\alpha}_1(x), ---, \hat{\alpha}_k(x), \{\frac{\partial \log p(x/\underline{\alpha})}{\partial \alpha_1}\}^*, ---, \{\frac{\partial \log p(x/\underline{\alpha})}{\partial \alpha_k}\}^*$ is described by [4,5]

$$\begin{bmatrix} V(\underline{\alpha}) & \Delta(\underline{\alpha}) \\ \Delta^*(\underline{\alpha}) & I(\underline{\alpha}) \end{bmatrix} \tag{2.3}$$

where $V(\underline{\alpha})$, $I(\underline{\alpha})$, $\Delta(\underline{\alpha})$ are k x k submatrices defined by

$$V_{ij}(\underline{\alpha}) = \underset{x}{E}[(\hat{\alpha}_i(x) - \alpha_i - b_i(\underline{\alpha}))(\hat{\alpha}_j(x) - \alpha_j - b_j(\underline{\alpha}))^*] \tag{2.4}$$

$$I_{ij}(\underline{\alpha}) = \underset{x}{E}[(\frac{\partial \log p(x/\underline{\alpha})}{\partial \alpha_i})^* \frac{\partial \log p(x/\underline{\alpha})}{\partial \alpha_j}] \tag{2.5}$$

$$\Delta_{ij}(\underline{\alpha}) = \underset{x}{E}[(\hat{\alpha}_i(x) - \alpha_i - b_i(\underline{\alpha})) \frac{\partial \log p(x/\underline{\alpha})}{\partial \alpha_j}]$$

$$= \delta_{ij} + \frac{\partial b_i(\underline{\alpha})}{\partial \alpha_j} \quad , \quad i,j = 1,2, ---, k \tag{2.6}$$

The matrix of Eq. (2.3) is a nonnegative definite Hermitian matrix. It can be shown that $V(\underline{\alpha}) - \Delta(\underline{\alpha}) I^{-1}(\underline{\alpha}) \Delta^*(\underline{\alpha})$ is also a non-negative definite Hermitian matrix, where $I^{-1}(\underline{\alpha})$ is the inverse of $I(\underline{\alpha})$.

By considering only the diagonal element, we have

$$\underset{x}{\mathrm{var}}[\hat{\alpha}_i(x)] = V_{ii}(\underline{\alpha}) \geq \sum_{m=1}^{k} \sum_{n=1}^{k} (\delta_{im} + \frac{\partial b_i(\underline{\alpha})}{\partial \alpha_m})[I^{-1}(\underline{\alpha})]_{m,n} (\delta_{in} + \frac{\partial b_i(\underline{\alpha})}{\partial \alpha_n})^* \tag{2.7}$$

Therefore the mean square error of the estimator $\hat{\alpha}(x)$ satisfies

$$E\,|\hat{\alpha}_i(x) - \alpha_i|^2 \geq |b_i(\underline{\alpha})|^2 + \sum_{m=1}^{k} \sum_{n=1}^{k} (\delta_{im} + \frac{\partial b_i(\underline{\alpha})}{\partial \alpha_m})[I^{-1}(\underline{\alpha})]_{mn} (\delta_{in} + \frac{\partial b_i(\underline{\alpha})}{\partial \alpha_n})^* \tag{2.8}$$

The equality in Eqs. (2.7) and (2.8) holds only when

$$\hat{\alpha}_i(x) - \alpha_i - b_i(\underline{\alpha}) = \sum_{j=1}^{k} K_{ij}(x) (\frac{\partial \log p(x/\underline{\alpha})}{\partial \alpha_j})^* \tag{2.9}$$

for some set of functions $K_{i1}(x), ---, K_{ik}(x)$. If the $\hat{\alpha}_i(x)$ is an unbiased estimator, Eqs. (2.7) and (2.8) reduce to

$$\underset{x}{E}\,|\hat{\alpha}_i(x) - \alpha_i|^2 = \underset{x}{\mathrm{var}} \, \hat{\alpha}_i(x) \geq [I^{-1}(\underline{\alpha})]_{i,i} \tag{2.10}$$

The equality in Eq. (2.10) holds only when

$$\hat{\alpha}_i(x) - \alpha_i = \sum_{j=1}^{k} K_{ij}(x) \{\frac{\partial \log p(x/\underline{\alpha})}{\partial \alpha_j}\}^* \tag{2.11}$$

3. UNCONDITIONAL ESTIMATE OF MULTIPARAMETERS

In the previous sections it was assumed that no a priori knowledge of parameters was available. However, in many cases the observer has some a priori knowledge concerning the value of the unknown parameters. Under these circumstances, the estimate should be unconditional [1].

Let $\check{\alpha}(x)$ be any unconditional estimate of a single unknown parameter α, and let

$$\underset{x}{E}[\check{\alpha}(x)] = \int_X \check{\alpha}(x) p(x/\alpha) dx = \alpha + b(\alpha) \tag{3.1}$$

Then it can be shown [12] that the variance of this estimate satisfies the following inequality under some regularity conditions:

$$\underset{\alpha,x}{\mathrm{var}} \, \check{\alpha}(x) = \int_\Omega \int_X |\check{\alpha}(x) - \alpha - b(\alpha)|^2 p(x/\alpha) \sigma(\alpha) dx d\alpha$$

$$\geq \frac{|1 + \underset{\alpha}{E} \, b'(\alpha)|^2}{\underset{\alpha}{E}[I(\alpha)]} \tag{3.2}$$

The equality of Eq. (3.2) holds only when

$$\check{\alpha}(x) - \alpha - b(\alpha) = K \left\{\frac{\partial \log p(x/\alpha)}{\partial \alpha}\right\}^* \tag{3.3}$$

for all $x \epsilon X$, $\alpha \epsilon \Omega$ (the parameter space), and for some constant K.

Now the mean square error between $\check{\alpha}(x)$ and the true value of α is given by[1]

$$\underset{\alpha}{E}\ \underset{x}{E} \left| \check{\alpha}(x) - \alpha \right|^2 \geq \underset{\alpha}{E} \left|b(\alpha)\right|^2 + \frac{\left| 1 + \underset{\alpha}{E}\, b'(\alpha) \right|^2}{\underset{\alpha}{E}\ I(\alpha)} \tag{3.4}$$

In the particular case when $\check{\alpha}(x)$ is unbiased for all α, then $b(\alpha) = 0$, and Eqs.(3.2) and (3.4) will reduce to

$$\underset{\alpha}{E}\ \underset{x}{E} \left|\check{\alpha}(x) - \alpha\right|^2 = \underset{\alpha,x}{\mathrm{var}}[\check{\alpha}(x)] \geq \frac{1}{\underset{\alpha}{E}\ I(\alpha)} \tag{3.5}$$

Now let us consider the multiparameter case. Let $\underline{\alpha} = [\alpha_1, ---, \alpha_k]$ be the set of unknown parameters to be estimated based on the probability density function $p(x/\underline{\alpha})$ and a priori probability density function $\sigma(\underline{\alpha})$. The discussion of the previous section can be applied in an analogous fashion, and the following conclusion will be obtained: let $\check{\alpha}_i(x)$ be an unconditional estimate of α_i such that

$$\int \check{\alpha}_i(x) p(x/\underline{\alpha}) dx = \alpha_i + b_i(\underline{\alpha}) \quad , \quad i = 1, ---, k \tag{3.6}$$

Then the k x k matrix $V - \Delta\, I^{-1} \Delta^*$ is a non-negative definite Hermitian matrix, where V, I and Δ are k x k matrices such that

$$V_{ij} = \underset{\underline{\alpha}}{E}\ V_{ij}(\underline{\alpha}) \quad , \quad i,j = 1, ---, k \tag{3.7}$$

$$I_{ij} = \underset{\underline{\alpha}}{E}\ I_{ij}(\underline{\alpha}) \tag{3.8}$$

$$\Delta_{ij} = \underset{\underline{\alpha}}{E}\ \Delta_{ij}(\underline{\alpha}) \tag{3.9}$$

and where $V_{ij}(\underline{\alpha})$, $I_{ij}(\underline{\alpha})$ and $\Delta_{ij}(\underline{\alpha})$ have been defined by Eqs. (2.4), (2.5), and (2.6). Considering only the diagonal elements of the matrix $V - \Delta\, I^{-1} \Delta^*$, we have

$$\underset{\underline{\alpha},x}{\mathrm{var}}[\check{\alpha}_i(x)] = \underset{\underline{\alpha}}{E}\ \underset{x}{E} \left| \check{\alpha}_i(x) - \alpha_i - b_i(\underline{\alpha}) \right|^2$$

$$\geq \sum_{m=1}^{k} \sum_{n=1}^{k} \left(\delta_{im} + \underset{\underline{\alpha}}{E} \frac{\partial b_i(\underline{\alpha})}{\partial \alpha_m}\right) [I^{-1}]_{mn} \left(\delta_{in} + \underset{\underline{\alpha}}{E} \frac{\partial b_i(\underline{\alpha})}{\partial \alpha_n}\right)^* \tag{3.10}$$

---

1. Middleton [1] and Hancock and Wintz [3] also give a lower bound for the error of unconditional estimate which is quite different from Eqs. (3.2) and (3.4):

   [1] page 943, Equation (21.6)
   [3] page 123, Equation (5-13)

The mean square error of the estimate $\check{\alpha}_i(x)$ satisfies

$$\underset{\underline{\alpha}}{E}\ \underset{x}{E} \left| \check{\alpha}_i(x) - \alpha_i \right|^2 \geq \underset{\underline{\alpha}}{E} \left|b_i(\underline{\alpha})\right|^2 + \sum_{m=1}^{k} \sum_{n=1}^{k} \left(\delta_{im} + \underset{\underline{\alpha}}{E} \frac{\partial b_i(\underline{\alpha})}{\partial \alpha_m}\right) [I^{-1}]_{mn} \left(\delta_{in} + \underset{\underline{\alpha}}{E} \frac{\partial b_i(\underline{\alpha})}{\partial \alpha_n}\right)^* \tag{3.11}$$

The equality in Eqs.(3.10) and (3.11) holds only when

$$\check{\alpha}_i(x) - \alpha_i - b_i(\underline{\alpha}) = \sum_{j=1}^{k} K_{ij} \left(\frac{\partial \log p(x/\underline{\alpha})}{\partial \alpha_j}\right)^* \tag{3.12}$$

for all $\underline{\alpha} \epsilon \Omega$, all $x \epsilon X$, and for some set of constants $K_{i1}, ---, K_{ik}$. In the case of unbiased estimators, Eqs. (3.10) and (3.11) reduce to

$$\underset{\underline{\alpha}}{E}\ \underset{x}{E} \left| \check{\alpha}_i(x) - \alpha_i \right|^2 = \underset{\underline{\alpha},x}{\mathrm{var}} [\check{\alpha}_i(x)] \geq [I^{-1}]_{ii} \tag{3.13}$$

4. AN APPLICATION: JOINT ESTIMATION OF RADAR PARAMETERS

Application of the conventional Cramer-Rao inequality, Eq.(1.6), to the estimation of radar parameters is not new [6,7]. Based on these previous works, we attempt to formulate a more general treatment of the radar parameter estimation problem using Eq.(3.13). In particular (a) a prior knowledge concerning unknown parameters is taken into consideration, (b) no limitation is placed on the number of parameters simultaneously estimated, and (c) it is not required that the power spectrum of the additive noise be bandlimited or flat.

The relation between this approach and the method of maximum likelihood adopted by Bello [8] and Kelly et. al. [9,10] will also be discussed. Assume that a given signal is transmitted periodically and that M returned echoes are observed. Assume also that the radial velocity of the moving target is constant and that the variation of the target range can be ignored. The i-th observed data is given by

$$x_i(t) = \alpha_i w(t-\tau) \exp(j 2\pi\lambda t) + n_i(t) \quad , \ i = 1, ---, M \tag{4.1}$$

where α_i is the signal amplitude, τ the time delay, and λ the doppler-shift frequency. The quantity α is a complex Gaussian random variable [12], if the magnitude is Rayleigh distributed and the phase is uniformly distributed between 0 and 2π. Eq.(4.1) can be written as

$$X(t) = \underline{\alpha}\, w(t-\tau) \exp(j 2\pi\lambda t) + N(t) \triangleq S(t;\underline{\alpha},\tau,\lambda) + N(t) \tag{4.2}$$

where

$$X(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \cdot \\ \cdot \\ \cdot \\ x_M(t) \end{bmatrix}, \quad \underline{\alpha} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \cdot \\ \cdot \\ \cdot \\ \alpha_M \end{bmatrix}, \quad N(t) = \begin{bmatrix} n_1(t) \\ n_2(t) \\ \cdot \\ \cdot \\ \cdot \\ n_M(t) \end{bmatrix} \tag{4.3}$$

The noise process N(t) is assumed to be an M-variate complex-valued stationary Gaussian analytic process with

$$E[N(t)] = 0 \text{ and } E[N(s)N^*(t)] = R(s-t).$$

The likelihood ratio is well defined and is given by

$$\Lambda(X/\underline{\alpha},\tau,\lambda,) = \exp\left\{[S(\cdot),X(\cdot)]_{H(R)} + [X(\cdot),S(\cdot)]_{H(R)} - [S(\cdot),S(\cdot)]_{H(R)}\right\} \quad (4.4)$$

where [,] is the reproducing inner product on the reproducing kernel Hilbert space generated by the covariance matrix function R(s-t) [12].

Assume that the covariance matrix function is separable so that

$$R(s-t) = R_N\, r(s-t) \quad (4.5)$$

where R_N is an M x M constant matrix and r(s-t) is a non-negative definite scalar function. Then the reproducing inner product can be written as

$$[F(\cdot),G(\cdot)]_{H(R)} = \sum_{i=1}^{M}\sum_{j=1}^{M} [f_i(\cdot),g_j(\cdot)]_{h(r)} [R_N^{-1}]_{i,j} \quad (4.6)$$

where $f_i(t)$ is the i-th element in an M-dimensional function F(t). The inner product $[\ ,\ ]_{h(r)}$ is the reproducing inner product on the reproducing kernel Hilbert space generated by r(s-t).

Let us restrict our attention to unbiased estimators which simultaneously estimate M+2 unknown parameters: α_1, - - - α_M , τ, and λ. In Eq.(2.5) set $\alpha_i=\alpha_i$, i=1, - - - ,M; $\alpha_{M+1}=\tau$; and $\alpha_{M+2}=\lambda$. After straightforward computation, the entries of Fisher's information matrix I are obtained as follows:

$$I_{ij}(\underline{\alpha},\tau,\lambda) = \|w(t-\tau)\exp(j2\pi\lambda t)\|^2_{h(r)} [R_N^{-1}]_{ij} \quad , i,j = 1, - - - ,M \quad (4.7)$$

$$I_{\tau\tau}(\underline{\alpha},\tau,\lambda) = 2\,\|\dot{w}(t-\tau)\exp(j2\pi\lambda t)\|^2_{h(r)}\ \underline{\alpha}^* R_N^{-1}\underline{\alpha} \quad (4.8)$$

$$I_{i\tau}(\underline{\alpha},\tau,\lambda) = -\,[w(t-\tau)\exp(j2\pi\lambda t),\dot{w}(t-\tau)\exp(j2\pi\lambda t)]_{h(r)} \cdot [R_N^{-1}]_{ij}\ \alpha_j \quad (4.9)$$

$$I_{\lambda\lambda}(\underline{\alpha},\tau,\lambda) = 2\,\|j2\pi t\ w(t-\tau)\exp(j2\pi\lambda t)\|^2_{h(r)}\underline{\alpha}^* R_N^{-1}\underline{\alpha} \quad (4.10)$$

$$I_{i\lambda}(\underline{\alpha},\tau,\lambda) = [w(t-\tau)\exp(j2\pi\lambda t),j2\pi t\ w(t-\tau)\exp(j2\pi\lambda t)]_{h(r)}[R_N^{-1}]_{ij}\ \underline{\alpha}_j \quad (4.11)$$

$$I_{\tau\lambda}(\underline{\alpha},\tau,\lambda) = 2\,\mathrm{Re}[-\dot{w}(t-\tau)\exp(j2\pi\lambda t),j2\pi t\ w(t-\tau)\exp(j2\pi\lambda t)]_{h(r)}\ \underline{\alpha}^* R_N^{-1}\underline{\alpha} \quad (4.12)$$

Let the a priori distribution function of the complex Gaussian multivariate $\underline{\alpha}$ be

$$\sigma(\underline{\alpha})d\underline{\alpha}d\underline{\alpha}^* = \frac{1}{(2\pi)^M \det R_S}\exp(-\underline{\alpha}^* R_S^{-1}\underline{\alpha})\ d\underline{\alpha}d\underline{\alpha}^* \quad (4.13)$$

where $R_S=E[\underline{\alpha}\,\underline{\alpha}^*]$ and $d\underline{\alpha}d\underline{\alpha}^* = \prod_{i=1}^{M} d\alpha_i d\alpha_i^*$. The a priori probability functions of τ and λ are not specified here. Instead we impose the following condition:

$$\|w(t-\tau)\exp(j2\pi\lambda t)\|^2_{h(r)} = \mu \quad ;\ \text{a constant} \quad (4.14)$$

This in turn specifies the admissable range of τ and λ.

After some manipulation, the averaged Fisher's information matrix is obtained as follows:

$$I_{ij} = \mu\,[R_N^{-1}]_{ij} \quad , \quad i,j = 1, - - - ,M \quad (4.15)$$

$$I_{\tau\tau} = 8\pi^2 B_e^2 \mu\ \mathrm{trace}\,[R_S R_N^{-1}] \quad (4.16)$$

$$I_{i\tau} = 0 \quad , \ i = 1, - - - ,M \quad (4.17)$$

$$I_{\lambda\lambda} = 8\pi^2 T_e^2 \mu\ \mathrm{trace}\,[R_S R_N^{-1}] \quad (4.18)$$

$$I_{i\lambda} = 0 \quad , \quad i = 1, - - - ,M \quad (4.19)$$

and

$$I_{\tau\lambda} = 8\pi^2\ T_e B_e \mu\ \mathrm{trace}\,[R_S R_N^{-1}] \quad (4.20)$$

where B_e and T_e are the effective bandwidth and duration of the signal waveform w(t), defined by

$$T_e^2 = \frac{\|t\ w(t)\|^2_{h(r)}}{4\pi^2\,\|w(t)\|^2_{h(r)}} \quad (4.21)$$

and

$$B_e^2 = \int_{-\infty}^{\infty}\frac{|f\ W(f)|^2}{P(f)}\,df \Big/ 4\pi^2\int_{-\infty}^{\infty}\frac{|W(f)|^2}{P(f)}\,df \quad (4.22)$$

Here W(f) and P(f) are the Fourier transforms of the signal w(t) and the covariance function r(s-t), respectively. The quantity ρ is defined by

$$\rho = \frac{\mathrm{Re}[\dot{w}(t),\ j2\pi t\ w(t)]_{h(r)}}{\|\dot{w}(t)\|_{h(r)}\ \|2\pi t\ w(t)\|_{h(r)}} \quad (4.23)$$

Then the averaged information matrix is

$$I = \begin{bmatrix} I(\underline{\alpha}) & \underline{0} \\ \underline{0} & I(\tau,\lambda) \end{bmatrix} \quad (4.24)$$

where the submatrices $I(\underline{\alpha})$ and $I(\tau,\lambda)$ are given by

$$I(\underline{\alpha}) = \mu R_N^{-1} \tag{4.25}$$

and

$$I(\tau,\lambda) = 8\pi^2\mu \ \text{trace}\ [R_S R_N^{-1}] \begin{bmatrix} B_e^2 & \rho T_e B_e \\ \rho T_e B_e & T_e^2 \end{bmatrix} \tag{4.26}$$

On inverting I and considering only the diagonal terms, we obtain

$$\underset{\underline{\alpha}}{E}\ \underset{x}{E} \mid \check{\alpha}_i(X) - \alpha_i \mid^2 > \frac{[R_N]_{ii}}{\mu}, \quad i = 1,2,---,M \tag{4.27}$$

$$\underset{\underline{\alpha}}{E}\ \underset{x}{E} \mid \check{\tau}(X) - \tau \mid^2 \geq \frac{1}{8\pi^2 B_e^2(1-\rho^2)\ \mu\ \text{trace}\ [R_S R_N^{-1}]} \tag{4.28}$$

and

$$\underset{\underline{\alpha}}{E}\ \underset{x}{E} \mid \check{\lambda}(X) - \lambda \mid^2 \geq \frac{1}{8\pi^2 T_e^2(1-\rho^2)\ \mu\ \text{trace}\ [R_S R_N^{-1}]} \tag{4.29}$$

5. UNCONDITIONAL MAXIMUM LIKELIHOOD ESTIMATION

In the previous section, a lower bound for the mean square error of unbiased estimation of radar parameters have been calculated by use of the generalized Cramer-Rao inequality. The derivation of these results, however, indicates neither the existence of the minimum variance unbiased estimator nor its structure.

Maximum likelihood estimation, when a priori knowledge unknown parameters is available, is called unconditional maximum likelihood estimation [1]. This is equivalent to the method of inverse probability used by Woodward and others[11,8]. In this section the structure of unconditional maximum likelihood estimators (UMLE) is obtained and it will be shown that their error variances approach the ultimate accuracy given by Eqs.(4.27)-(4.29) as the output signal-to-noise ratio increases without bound.

If the a priori probability of τ and λ is denoted by $\sigma(\tau,\lambda)$, the UML estimate is the set of $\underline{\alpha}$, τ, and λ which maximizes the quantity

$$L(X;\underline{\alpha},\tau,\lambda) = \Lambda(X/\underline{\alpha},\tau,\lambda)\ \sigma(\underline{\alpha})\sigma(\tau,\lambda) \tag{5.1}$$

or, equivalently, the quantity

$$\log \Lambda(X/\underline{\alpha},\tau,\lambda)\ \sigma(\underline{\alpha})\ \sigma(\tau,\lambda) = -[\underline{\alpha}-(R_S^{-1}+\mu R_N^{-1})^{-1} R_N^{-1}\underline{\varepsilon}]^*[R_S^{-1}+\mu R_N^{-1}]$$
$$[\underline{\alpha}-(R_S^{-1}+\mu R_N^{-1})^{-1} R_N^{-1}\ \underline{\varepsilon}] + \underline{\varepsilon}^* R_N^{-1}\ [R_S^{-1}+\mu R_N^{-1}]^{-1}\ R_N^{-1}\underline{\varepsilon} + \log(2\pi)^M \det R_S \tag{5.2}$$

where

$$\underline{\varepsilon} = \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \cdot \\ \cdot \\ \varepsilon_M \end{bmatrix}, \text{ and } \varepsilon_i = [w(t-\tau)e^{j2\pi\lambda t}, x_i(t)]_{h(r)},\ i=1,--,M \tag{5.3}$$

Since $\underline{\varepsilon}$ is a Gaussian multivariate depending on τ and λ, the UML estimate $\check{\tau}(X)$ and $\check{\lambda}(X)$ are those that maximize

$$\underline{\varepsilon}^*\ G\ \underline{\varepsilon} + \log \sigma(\tau,\lambda) \tag{5.4}$$

where

$$G = R_N^{-1}[R_S^{-1} + \mu R_N^{-1}]^{-1} R_N^{-1} \tag{5.5}$$

The UML estimate $\check{\alpha}(X)$ is given by

$$\underline{\check{\alpha}}(X) = [R_S^{-1} + \mu R_N^{-1}]^{-1} R_N^{-1}\underline{\check{\varepsilon}} \tag{5.6}$$

where

$$\check{\varepsilon}_i = [w(t-\check{\tau})e^{j2\pi\check{\lambda}t}, x_i(t)]_{h(r)}, \quad i = 1,2,---,M$$

Let us consider the case where the output signal-to-noise ratio is extremely high [8,9,10]. In this case, the second term of Eq.(5.4) is negligible and the first term is given approximately by

$$\underline{\varepsilon}^* G\ \underline{\varepsilon} \approx \frac{\underline{\alpha}^* R_N^{-1}\underline{\alpha}}{\mu} |\chi_w(\check{\tau}-\tau,\check{\lambda}-\lambda)|^2 \tag{5.7}$$

where $\chi_w(\tau,\lambda)$ is a generalized ambiguity function of the signal waveform $w(t)$ and is defined by

$$\chi_w(\check{\tau}-\tau,\check{\lambda}-\lambda) = [w(t-\check{\tau})e^{j2\pi\check{\lambda}t}, w(t-\tau)e^{j2\pi\lambda t}]_{h(r)} \tag{5.8}$$

If $r(s-t) = \delta(s-t)$, the χ_w reduces to the conventional ambiguity function [7].

Assuming the existence of Taylor's series expansion of $|\chi_w|^2$ around (τ,λ) and using the quantities T_e, B_e, and ρ defined in the previous section, we obtain [12]

$$\exp(\underline{\varepsilon}^*\underline{\varepsilon}) \cong k \exp\left\{-4\pi^2\mu\underline{\alpha}^* R_N^{-1}\underline{\alpha} \begin{bmatrix} \check{\tau}-\tau \\ \check{\lambda}-\lambda \end{bmatrix}' \begin{bmatrix} \rho B_e^2 & B_e T_e \\ \rho B_e T_e & T_e^2 \end{bmatrix} \begin{bmatrix} \check{\tau}-\tau \\ \check{\lambda}-\lambda \end{bmatrix}\right\} \tag{5.9}$$

Since the left side of Eq.(5.9) is proportional to the a posteriori probability of τ and λ, we have

$$E[\check{\tau}(X)] = \tau, \quad E[\check{\lambda}(X)] = \lambda \tag{5.10}$$

and

$$\text{Var}[\check{\tau}(X)] = \frac{1}{8\pi^2 B_e^2(1-\rho^2)\mu\ \underline{\alpha}^* R_N^{-1}\underline{\alpha}} \tag{5.11}$$

$$\text{var}[\check{\lambda}(X)] = \frac{1}{8\pi^2 T_e^2(1-\rho^2)\mu\ \underline{\alpha}^* R_N^{-1}\underline{\alpha}} \tag{5.12}$$

where $\underline{\alpha}$ is the true value of the signal amplitude under observation. The right sides of Eqs.(4.28) and (4.29) are obtained when the denominators of Eqs.(5.11) and (5.12) are averaged with respect to $\underline{\alpha}$, since

$E[\underline{\alpha}^* R_N^{-1}\underline{\alpha}] = \text{trace}[R_S R_N^{-1}]$. After τ and λ are estimated, the UML estimate of $\underline{\alpha}$ is given by Eq.(5.6).

It can be shown [12] that this UMLE $\underline{\check{\alpha}}(X)$ satisfies the condition of Eq.(3.12) and therefore possesses minimum variance, i.e. from Eqs. (3.10) and (3.11)

$$\mathrm{var}[\check{\alpha}_i(X)] = [(I + \mu R_S R_N^{-1})\ \mu R_S R_N^{-1} R_S (I + \mu R_S R_N^{-1})^{-1}]_{ii} \tag{5.13}$$

and

$$\underset{\alpha}{E}\ \underset{x}{E}\ |\check{\alpha}_i(X) - \alpha_i|^2 = [R_S(I+\mu R_S R_N^{-1})^{-1}]_{ii} \tag{5.14}$$

If the output signal-to-noise ratio is large enough so that the minimum eigenvalue of $\mu R_S R_N^{-1} >> 1$, then Eq.(5.14) becomes

$$\underset{\alpha}{E}\ \underset{x}{E}\ |\check{\alpha}_i(X) - \alpha_i|^2 = \frac{[R_N]_{ii}}{\mu} \tag{5.15}$$

This last expression is equal to the right side of Eq.(4.27).

Therefore, as the signal-to-noise ratio increases without bound, the UMLE $\check{\alpha}(X)$ becomes unbiased and approaches the ultimate lower bound given by the Cramer-Rao inequality.

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APPENDIX
DERIVATION OF EQ.(3.2)

From Eq.(3.1) we have

$$\alpha + b(\alpha) = \int \check{\alpha}(x)p(x/\alpha)dx \tag{A.1}$$

Since $p(x/\alpha)$ is a probability density, it follows that

$$1 = \int_X p(x/\alpha)dx \tag{A.2}$$

Assuming the regularity condition which permits us to differentiate with respect to α inside the integration with respect to x, we have from Eqs.(A-1) and (A-2).

$$1 + b'(\alpha) = \int_X \check{\alpha}(x)\ \frac{\partial \log p(x/\alpha)}{\partial\alpha}\ p(x/\alpha)dx \tag{A-3}$$

and

$$0 = \int_X \frac{\partial \log p(x/\alpha)}{\partial\alpha}\ p(x/\alpha)dx \tag{A-4}$$

Let $f(\alpha)$ be an arbitrary function of α. On multiplying Eq.(A-4) by $f(\alpha)$ and subtracting the result from Eq.(A-3), we have

$$1 + b'(\alpha) = \int_X (\check{\alpha}(x) - f(\alpha))\ \frac{\partial \log p(x/\alpha)}{\partial\alpha}\ p(x/\alpha)dx \tag{A-5}$$

On taking expectation with respect to α, we find

$$1 + \underset{\alpha}{E}\ b'(\alpha) = \int_\Omega \int_X (\check{\alpha}(x) - f(\alpha))\ \frac{\partial \log p(x/\alpha)}{\partial\alpha}\ p(x/\alpha)\ \sigma(\alpha)dxd\alpha \tag{A-6}$$

By use of the Cauchy-Schwartz inequality, we obtain

$$|1 + \underset{\alpha}{E}\ b'(\alpha)|^2 \leq \int_\Omega \int_X |\check{\alpha}(x) - f(\alpha)|^2\ p(x/\alpha)\ \sigma(\alpha)dxd\alpha \cdot \int_\Omega \int_X \left|\frac{\partial \log p(x/\alpha)}{\partial\alpha}\right|^2 p(x/\alpha)\ \sigma(\alpha)dxd\alpha \tag{A-7}$$

Where the equality holds if and only if

$$\check{\alpha}(x) - f(\alpha) = K \left\{\frac{\partial \log p(x/\alpha)}{\partial\alpha}\right\}^* \tag{A-8}$$

for some constant K. After taking expectation with respect to x for a given α and using Eq.(A-4), the result is

$$f(\alpha) = \underset{x}{E}\ [\check{\alpha}(x)] = \alpha + b(\alpha) \tag{A-9}$$

i.e. Eq.(A-9) is a necessary (but not sufficient) condition for the equality of Eq.(A-7) to hold. Therefore the lower bound of the variance of the unconditional estimate is

$$\mathrm{var}\ [\check{\alpha}(x)] = \int_\Omega \int_X |\check{\alpha}(x) - \alpha - b(\alpha)|^2\ p(x/\alpha)\ \sigma(\alpha)dxd\alpha \geq \frac{|1 + \underset{\alpha}{E}\ b'(\alpha)|^2}{\underset{\alpha}{E}\ I(\alpha)} \tag{A-10}$$

Where the equality holds if and only if

$$\check{\alpha}(x) - \alpha - b(\alpha) = K \left\{\frac{\partial \log p(x/\alpha)}{\partial\alpha}\right\}^*$$

for all $x \epsilon X$ and $\alpha \epsilon \Omega$.