On Queueing Networks and Loss Networks *

Hisashi Kobayashi and Brian L. Mark †
Department of Electrical Engineering
Princeton University, Princeton, NJ 08544

Abstract

Recently, there has been an increasing interest in loss networks [5], since a loss network can provide a mathematical framework for the study of connection-oriented services [8]. In this paper, we present relations between a class of loss networks and queueing networks whose stationary distributions have a product form.

1 Introduction

The earliest work on queueing networks with product form goes back to J.R. Jackson's 1963 paper [4]. Driven by the practical applications to job-shops and computer systems, much research has been devoted to studying approximations, computational algorithms, and asymptotic behavior for Jackson networks and generalizations. More recently, there has been a renewed interest in generalizations of the loss models originally studied by Erlang (cf. [12]) in the context of telephone exchanges. Loss networks provide models for studying the blocking behavior of connection-oriented services in communication networks. New applications of loss networks include broadband optical networks and wireless networks.

The purpose of this paper is to show how some aspects of the theory developed for queueing networks can be carried over to the study of loss networks.

2 Generalized Loss Stations

We use the general term *station*, to denote an entity which provides service to arriving customers. A station consists of a number of *servers* and possibly a *waiting room*. A *loss station* is characterized by having a finite number of servers and no waiting room. An arriving customer either begins service immediately or is rejected due to lack of a sufficient number of available servers. By contrast, a *queueing station* has infinite waiting room; no customer is rejected.

The original loss model studied by Erlang is equivalent to an $M/M/B(0)$ queue (see Figure 1); i.e., a loss station

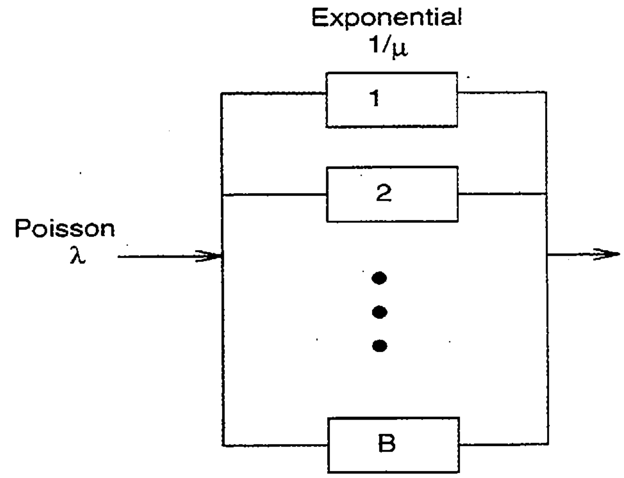

Figure 1: Erlang Model

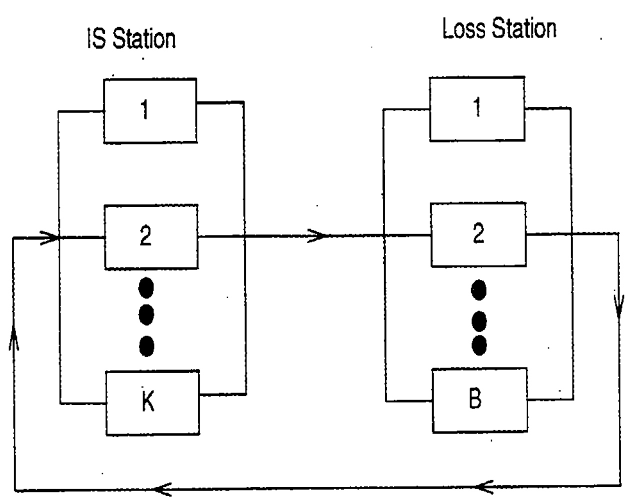

Figure 2: Engset Station

*This research has been supported, in part, by the New Jersey Commission on Science and Technology and the Ogasawara Foundation for Promotion of Science and Technology

† The second author has been supported by an NSERC postgraduate scholarship.

with B servers where arriving customers form a Poisson process with rate λ and each customer occupies a server for an exponentially distributed holding time with mean $1/\mu$. The stationary distribution of the number of *busy* servers is given by a truncated Poisson distribution:

$$P_B(n) = \frac{a^n}{n!}\left[\sum_{i=0}^{B}\frac{a^i}{i!}\right]^{-1}, \quad 0 \le n \le B \tag{1}$$

where $a = \lambda/\mu$ is the *offered load.* As $B \to \infty$, $P_B(n) \to a^n/n!$, which is the stationary distribution of an infinite-server (IS) station or $M/M/\infty$ queue. The stationary probability that all servers are busy is given by the celebrated Erlang loss formula

$$E(B,a) \triangleq P_B(B) = \frac{a^B}{B!}\left[\sum_{i=0}^{B}\frac{a^i}{i!}\right]^{-1} \tag{2}$$

In the original Engset loss model, which is equivalent to an $M(K)/M/B(0)$ queue, the arrival process is generated by a finite source model with K sources (see Figure 2). Each source consists of a customer who waits for an exponentially distributed inter-generation time, arrives at the station where it either acquires a server for an exponentially distributed holding time or is blocked, and a new cycle begins.

We generalize the classical models by introducing a set, C, of customer classes with multiple server acquisition and general holding times. Customers of class $c \in C$ seek to acquire A_c servers for a generally distributed holding time with mean $1/\mu_c$. The *generalized IS station*, denoted $\cdot/G_C * A_C/\infty$, has an infinite number of servers. The *generalized loss station*, denoted $\cdot/G_C * A_C/B(0)$, has B servers with no waiting room; an arriving customer of class c is *blocked* if there are less than A_c available servers.

We consider two multiclass source models. The *multiclass Poisson source*, denoted M_C, consists of independent Poisson processes with rates $\lambda_c, c \in C$. The *multiclass finite source*, denoted $G_C(K_C)$, consists of independent finite sources with populations K_c and generally distributed inter-generation times with means $1/\nu_c$, for each $c \in C$. With these source models, we obtain generalizations of the Erlang and Engset loss station, denoted by $M_C/G_C * A_C/B(0)$ and $G_C(K_C)/G_C * A_C/B(0)$, respectively.

The IS station may be viewed as a limiting case of the Erlang station as the number of servers $B \to \infty$. Both types of stations may be characterized by a state process $\mathbf{n}(t) = (n_c(t) : c \in \mathcal{C})$, where $n_c(t)$ denotes the number of class c customers in the station at time t. Let $\pi_B(\mathbf{n})$ denote the equilibrium state distribution when there are B servers. The set of feasible states is $\mathcal{S}(B) = \{\mathbf{n} \ge \mathbf{0} : \sum_{c \in C} A_c n_c \le B\}$. The *departure process* from the Erlang station includes both customers who have successfully completed service and those who are blocked. In Appendix A, we prove the following result:

Theorem 2.1 *The generalized Erlang station is quasireversible and its state-process* $\mathbf{n}(t)$ *is a reversible Markov process with stationary distribution given by*

$$\pi_B(\mathbf{n}) = \frac{1}{G(B)}\prod_{c \in C}\frac{a_c^{n_c}}{n_c!}, \quad \mathbf{n} \in \mathcal{S}(B) \tag{3}$$

where $a_c = \lambda_c/\mu_c$ *and* $G(B)$ *is the normalization constant defined by*

$$G(B) = \sum_{\mathbf{n} \in \mathcal{S}(B)}\prod_{c \in C}\frac{a_c^{n_c}}{n_c!} \tag{4}$$

The following theorem is a generalization of a result first reported by Cohen [2].

Theorem 2.2 *For the generalized Engset loss system,* $\mathbf{n}(t)$ *is a reversible Markov process with stationary distribution:*

$$\pi_B(\mathbf{n}, \mathbf{K}) = \frac{1}{G(B)}\prod_{c \in C}\binom{K_c}{n_c}\left(\frac{\nu_c}{\mu_c}\right)^{n_c}, \quad \mathbf{n} \in \mathcal{S}(B) \tag{5}$$

Proof. This loss system can be viewed as a two-station closed queueing network consisting of a (generalized) IS station in tandem with a loss station. With Poisson source models both stations are quasireversible and possess the insensitivity property with respect to service time distributions. Hence, the stationary distribution of the tandem connection has the form

$$\pi_O(\mathbf{n}^1, \mathbf{n}^2) \propto \pi_1(\mathbf{n}^1)\pi_2(\mathbf{n}^2) \tag{6}$$

where $\mathbf{n}^1$ and $\mathbf{n}^2$ represent the number of customers of each class at the IS station and the loss station, respectively, and π_i, $i = 1, 2$ denote the marginal distributions of the two stations. Hence,

$$\pi_O(\mathbf{n}^1, \mathbf{n}^2) \propto \prod_{c \in C}\frac{(\lambda_c/\mu_c)^{n_c^1}}{n_c^1!}\frac{(\lambda_c/\nu_c)^{n_c^2}}{n_c^2!} \tag{7}$$

By making the identification $\mathbf{K} - \mathbf{n} = \mathbf{n}^1$ and $\mathbf{n} = \mathbf{n}^2$ and applying the state truncation property with the closed network constraint $\mathbf{n}^1 + \mathbf{n}^2 = \mathbf{K}$, we obtain that the stationary distribution of the original system has the form

$$\pi(\mathbf{K}, \mathbf{n}) \propto \prod_{c \in C}\binom{K_c}{n_c}\left(\frac{\nu_c}{\mu_c}\right)^{n_c} \tag{8}$$

which, upon normalization, yields the result (5). Reversibility can be established by showing that the distribution given by (5) satisfies detailed balance. □

3 Loss Networks

Let us now classify the servers of a loss station into a set, $\mathcal{J}$, of server *types.* There are B_j servers of type $j \in \mathcal{J}$ and $\sum_{j \in \mathcal{J}} B_j = B$. An arriving class c customer seeks to simultaneously acquire A_{jc} servers of type j (see Figure 3). We denote such a loss station by $\cdot/G_C * A_{\mathcal{J},C}/B_{\mathcal{J}}(0)$. Letting $B_j \to \infty$ for each $j \in \mathcal{J}$, we obtain a generalized IS station with multiple server types.

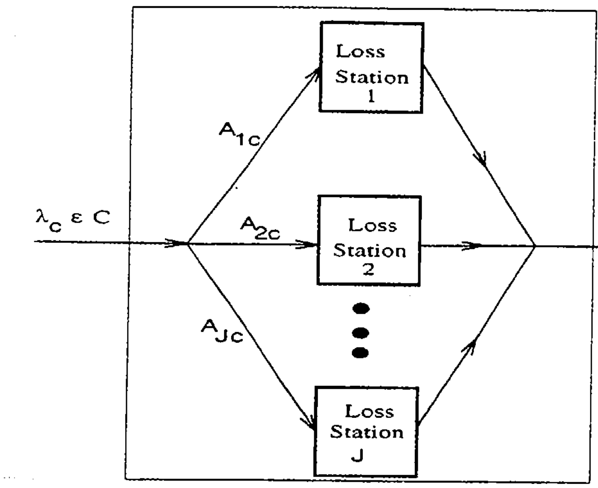

Figure 3: Generalized Loss Station with Multiple Server Types

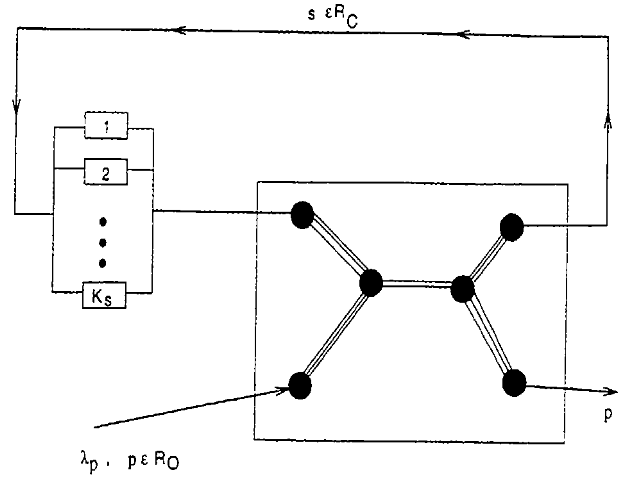

Figure 5: Mixed Loss Network

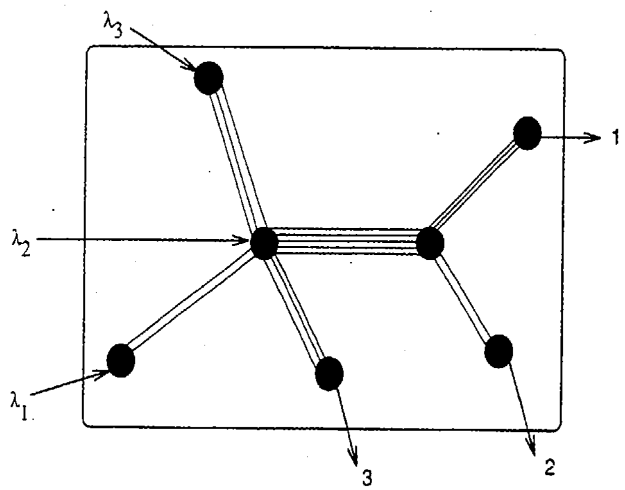

Figure 4: Open Loss Network

This station operation provides a model for the operation of a *circuit-switched network* with the following interpretation: each server represents a *circuit* and $\mathcal{J}$ is a set of links, with the jth consisting of B_j circuits. In keeping with standard notation for loss networks, we identify the set of call classes, $\mathcal{C}$, with the set of *routes*, $\mathcal{R}$. A route r call requires A_{jr} circuits on each link $j \in \mathcal{J}$. We then define a *loss network* (LN) as a $\cdot/G_{\mathcal{R}} * A_{\mathcal{J},\mathcal{R}}/B_{\mathcal{J}}(0)$ loss station. Letting the link capacities $B_j \to \infty$, we obtain an IS network (ISN), $\cdot/G_{\mathcal{R}} * A_{\mathcal{J},\mathcal{R}}/\infty$.

We introduce a source model for the LN station as follows. In analogy to *open* and *closed* subchains for customer routing in queueing networks (cf. [1, 11]), we classify the routes of a loss network into the set of open routes and the set of closed routes, denoted by R_O and R_C, respectively, with $\mathcal{R} = \mathcal{R}_O \cup \mathcal{R}_C$. The customer arrival process to an open route p is Poisson with rate λ_p, while that for a closed route s is a finite source of population K_s with mean inter-generation time $1/\nu_s$. All customer arrival processes are assumed to be independent. An *open loss network* (OLN), as depicted in Figure 4, consists entirely of open routes while a *closed loss network* consists entirely of closed routes. A *mixed loss network* (MLN), as shown in Figure 5, may have both kinds of routes and we denote it by $M_{\mathcal{R}_O}, G_{\mathcal{R}_C}(K_{\mathcal{R}_C})/G_{\mathcal{R}} * A_{\mathcal{J},\mathcal{R}}/B_{\mathcal{J}}(0)$. The MLN generalizes both the Erlang and Engset stations of the previous section. Denote its state process by $\mathbf{n}(t) = [\mathbf{n}_O(t), \mathbf{n}_C(t)]$, with $\mathbf{n}_O(t) = (n_p(t) : p \in \mathcal{R}_O)$ and $\mathbf{n}_C(t) = (n_s(t) : s \in \mathcal{R}_C)$. We have the following result for the MLN:

Theorem 3.1 *The state process of the mixed loss network is a reversible Markov process with equilibrium distribution given by*

$$\pi_{\mathbf{B}}(\mathbf{n}) = \frac{1}{G(\mathbf{B})} p_O(\mathbf{n}_O) p_C(\mathbf{n}_C), \qquad \mathbf{n} \in \mathcal{N}(\mathbf{B}) \tag{9}$$

where

$$p_O(\mathbf{n}_O) = \prod_{p \in \mathcal{R}_O} \frac{a_p^{n_p}}{n_p!}, \quad p_C(\mathbf{n}_C) = \prod_{s \in \mathcal{R}_C} \binom{K_s}{n_s} b_s^{n_s} \tag{10}$$

with $b_s = \nu_s/\mu_s, \quad s \in \mathcal{R}_c$,

$$\mathcal{N}(\mathbf{B}) = \{\mathbf{n} \geq \mathbf{0} : \mathbf{A}\mathbf{n} \leq \mathbf{B}\} \tag{11}$$

and

$$G(\mathbf{B}) = \sum_{\mathbf{n} \in \mathcal{N}(\mathbf{B})} p_O(\mathbf{n}_O) p_C(\mathbf{n}_C) \tag{12}$$

We can also show that an OLN is quasireversible. We omit the proofs of these results since they are straightforward generalizations of those in the previous section.

4 Properties of Loss Networks

4.1 Blocking Probabilities

With reference to the stationary distribution of the mixed loss network, the probability that the network state is such that no additional route r call can be accepted (i.e., the *time congestion* on route r) can be expressed in terms of the normalization constant $G(\mathbf{B})$ as:

$$\begin{aligned} L_r(\mathbf{B}) &= 1 - \sum_{\mathbf{n}\in\mathcal{N}(\mathbf{B}-\mathbf{A}_r)} \pi_{\mathbf{B}}(\mathbf{n}) \\ &= 1 - \frac{G(\mathbf{B}-\mathbf{A}_r)}{G(\mathbf{B})} \\ &= \frac{1}{G(\mathbf{B})} \sum_{\mathbf{n}\in\mathcal{N}_r(\mathbf{B})} p_O(\mathbf{n}_O)p_C(\mathbf{n}_C) \quad (13) \end{aligned}$$

where $\mathbf{A}_r$ is the rth column of $\mathbf{A}$ and $\mathcal{N}_r(\mathbf{B}) = \mathcal{N}(\mathbf{B})\backslash\mathcal{N}(\mathbf{B}-\mathbf{A}_r)$. The proportion of arriving calls on route $r \in \mathcal{R}$ which are rejected is the *call congestion* or *loss probability* for route p, denoted by $\tilde{L}_p$. By the PASTA property (cf. [14]), the distribution seen by the call arrivals on an open route $p \in \mathcal{R}_O$ is identical with the stationary distribution $\pi(\mathbf{n})$. Hence we have

$$L_p(\mathbf{B}) = \tilde{L}_p(\mathbf{B}), \quad \forall p \in \mathcal{R}_O \quad (14)$$

The proportion of arriving calls over a closed route $s \in \mathcal{R}_C$ that find the system in state $\mathbf{n}$ is

$$a_s(\mathbf{n},\mathbf{K}) = \frac{(K_s - n_s)\nu_s\pi_{\mathbf{B}}(\mathbf{n})}{\sum_{\mathbf{n}'\in\mathcal{N}(\mathbf{B})}(K_s - n'_s)\nu_s\pi_{\mathbf{B}}(\mathbf{n}')} \quad (15)$$

Using (9) and performing some algebraic manipulations yields the simple result

$$a_s(\mathbf{n},\mathbf{K}) = \pi_{\mathbf{B}}(\mathbf{n},\mathbf{K}_s^-) \quad (16)$$

where we have made the closed route populations explicit in the notation and $\mathbf{K}_s^-$ denotes the population vector $\mathbf{K}$ with one less source for route s. This is a generalization of the well-known property for the quasi-random input model (cf. [7]). The loss probability for route s calls is then given by

$$\tilde{L}_s(\mathbf{B}) = 1 - \frac{G(\mathbf{B}-\mathbf{A}_s,\mathbf{K}_s^-)}{G(\mathbf{B},\mathbf{K}_s^-)} \quad (17)$$

$$= \frac{1}{G(\mathbf{B},\mathbf{K}_s^-)} \sum_{\mathbf{n}\in\mathcal{N}_s(\mathbf{B},\mathbf{K}_s^-)} p_O(\mathbf{n}_O)p_C(\mathbf{n}_C,\mathbf{K}_s^-) \quad (18)$$

The expressions (13) and (18) may be viewed as generalized Erlang and Engset loss formulae, respectively.

4.2 Generating Functions

For the mixed loss network, the pgf, $\pi^*_{\mathbf{B}}(\mathbf{z}) = \sum_{\mathbf{n}\in\mathcal{N}(\mathbf{B})} \pi_{\mathbf{B}}(\mathbf{n})\mathbf{z}^{\mathbf{n}}$, is given by

$$\frac{1}{G(\mathbf{B})} \sum_{\mathbf{n}\in\mathcal{N}(\mathbf{B})} \prod_{p\in\mathcal{R}_O} \frac{(a_p z_p)^{n_p}}{n_p!} \prod_{s\in\mathcal{R}_C} \binom{K_s}{n_s} (b_s z_s)^{n_s} \quad (19)$$

where $\mathbf{z} = \prod_{r\in\mathcal{R}} z_r$. Letting $\mathbf{B} \to \infty$, we obtain the pgf of an IS network:

$$\pi^*_\infty(\mathbf{z}) = \exp\{\sum_{p\in\mathcal{R}_O} a_p(z_p - 1)\} \prod_{s\in\mathcal{R}_C} \left(\frac{1+b_s z_s}{1+b_s}\right)^{K_s}$$

From independence in the IS network, the pgf is the product of the marginal pgfs of the Poisson random variables n_p, $p \in \mathcal{R}_O$ and the binomial random variables n_s, $s \in \mathcal{R}_C$. From (19), we can obtain the marginal pgf, $\pi^*_p(z)$, for an open route p, by setting $z_r = 1$ for all $r \in \mathcal{R}\backslash\{p\}$ and $z_p = z$. Similarly, we can obtain the pgf, $\pi^*_s(z)$, corresponding to a closed route s.

The loss network can also be characterized by the link-state process $\mathbf{m}(t) = (m_j(t) : j \in \mathcal{J})$, where $m_j(t)$ is the number of occupied circuits on link j at time t. Clearly, $\mathbf{m}(t) = \mathbf{A}\mathbf{n}(t)$, so the stationary distribution of $\mathbf{m}(t)$ can be expressed as:

$$P_{\mathbf{B}}(\mathbf{m}) = \sum_{\mathbf{n}:\mathbf{A}\mathbf{n}=\mathbf{m}} \pi_{\mathbf{B}}(\mathbf{n}), \quad \mathbf{m} \in \mathcal{M}(\mathbf{B}) \quad (20)$$

where $\mathcal{M}(\mathbf{B}) = \{\mathbf{m} : \exists\mathbf{n} \in \mathcal{N}(\mathbf{B}), \mathbf{A}\mathbf{n} = \mathbf{m}\}$. Define the pgf of $P_{\mathbf{B}}(\mathbf{m})$ by $P^*_{\mathbf{B}}(\theta) = E[\theta^{\mathbf{m}}]$, where $\theta = \theta_1 \cdots \theta_J$. For the IS network, the pgf satisfies

$$P^*_\infty(\theta) \propto \exp(\sum_{p\in\mathcal{R}_O} a_p\theta^{\mathbf{A}_p}) \prod_{s\in\mathcal{R}_C} (1 + b_s\theta^{\mathbf{A}_s})^{K_s} \quad (21)$$

where $\theta^{\mathbf{A}_r} \triangleq \prod_{j\in\mathcal{J}} \theta_j^{A_{jr}}$.

Define the generating function (gf) of the normalization constant $G(\mathbf{B})$ with respect to the J-dimensional vector $\mathbf{B}$ by $G^*(\theta) = \sum_{\mathbf{B}\geq\mathbf{0}} G(\mathbf{B})\theta^{\mathbf{B}}$. It can be shown that $G^*(\theta)$ is given by

$$\left(\prod_{j=1}^{J} \frac{1}{1-\theta_j}\right) \exp\{\sum_{p\in\mathcal{R}_O} a_p\theta^{\mathbf{A}_p}\} \prod_{s\in\mathcal{R}_C} (1 + b_s\theta^{\mathbf{A}_s})^{K_s}$$

Using (21), we can write

$$G^*(\theta) \propto \left(\prod_{j=1}^{J} \frac{1}{1-\theta_j}\right) P^*_\infty(\theta) \quad (22)$$

4.3 Computation

The pgfs given in the previous section lead to several numerical solutions. The marginal distributions for an open and closed routes, respectively, can be obtained as:

$$\pi_p(n) = \frac{G(\mathbf{B} - n\mathbf{A}_p, \mathcal{R}\backslash\{p\})}{G(\mathbf{B},\mathcal{R})} \frac{a_p{}^n}{n!}, \quad 0 \leq n \leq N_p, \quad (23)$$

$$\pi_s(n) = \frac{G(\mathbf{B} - n\mathbf{A}_s, \mathcal{R}\backslash\{s\})}{G(\mathbf{B},\mathcal{R})} \binom{K_s}{n} b_s^n, \quad 0 \leq n \leq N_s \quad (24)$$

where $N_p = \min_{j\in\mathcal{J}}[B_j/A_{jp}]$, $p \in \mathcal{R}_O$ and $N_s = \min_{j\in\mathcal{J}}\{[B_j/A_{jr}], K_s\}$, $s \in \mathcal{R}_C$. The link-state probabilities satisfy, for each $j \in \mathcal{J}$, the following recurrence relation:

$$m_j P_{\mathbf{B}}(\mathbf{m}) = \sum_{p\in\mathcal{R}_O} A_{jp}a_p P_{\mathbf{B}}(\mathbf{m} - \mathbf{A}_p)$$

$$+ \sum_{s\in\mathcal{R}_C} A_{js}K_s b_s \sum_{k=1}^{K_s} (-b_s)^{k-1} P_{\mathbf{B}}(\mathbf{m} - k\mathbf{A}_s) \quad (25)$$

A recursive formula similar to (25) has also been obtained in the context of the more general BPP arrival processes [3, 10].

As in queueing networks, many of the parameters of interest for loss networks are expressed in terms of the normalization constant $G(\mathbf{B}, \mathbf{K})$. Likewise, a straightforward term-by-term summation is impractical, since the number of terms grows exponentially with the problem size. For an ISN, the normalization constant is given by

$$G(\infty, \mathbf{K}) = \exp\{\sum_{p \in \mathcal{R}_O} a_p\} \prod_{s \in \mathcal{R}_c} (1 + b_s)^{K_s} \quad (26)$$

$$= e^a \prod_{s \in \mathcal{R}_C} (1 + b_s)^{K_s} \quad (27)$$

where $a = \sum_{p \in \mathcal{R}_O} a_p$. In a *light traffic* regime, where the capacities B_j are sufficiently large relative to the offered loads, $G(\infty, \mathbf{K})$ can be used as an approximation to $G(\mathbf{B}, \mathbf{K})$ in the loss formulae given above.

For the generalized Engset station, Kogan [9] has investigated the approach of inverting the pgf $G^*(\theta)$ and studying the asymptotic properties of the normalization constant via the saddle-point method. Several authors [5, 13] have proposed a *reduced load approximation*, whereby blocking events on different links are assumed independent. The computation of blocking probabilities then reduces to iterative evaluations of the Erlang formula with modified offered loads.

5 Conclusion

We formalized several generalizations of the classical loss models studied by Erlang and Engset. Under multiclass service, with multiple server acquisition, the generalized loss stations retain the properties of insensitivity to holding time distributions and inter-generation time distributions for the finite source model. By letting the number of servers approach infinity, corresponding generalized IS queueing stations were obtained.

We showed that the notion of loss network arises from introducing server types as links in a circuit-switched network and interpreting the customer classes as routes. By considering Poisson and finite source models, we introduced the notions of open, closed, and mixed loss networks, in analogy to the corresponding concepts for queueing networks. We obtained expressions for time and call congestion for open and closed routes. Taking an approach similar to [11], we obtained generating functions which led to computational expressions for several quantities of interest.

Future work should lead to the development of better computational and asymptotic solution methods, as we have seen in the area of queueing networks in the past three decades.

A Proof of Theorem 2.1

We shall assume that the holding time distribution of class c customers is represented exactly, or approximately by one having a rational Laplace-Stieltjes transform (LST), $\Phi_c(s)$. The theorem is, in fact, true for abritrary distributions, but the proof would require us to consider continuous state-space Markov processes.

Using Cox's method of stages (cf. [7]), any rational LST can be expressed in the form

$$\Phi(s) = b_0 + \sum_{\phi=1}^{d} a_0 \cdots a_{\phi-1} b_\phi \prod_{i=1}^{\phi} \frac{\mu_i}{s + \mu_i} \quad (28)$$

where d is the number of *stages*, $a_i + b_i = 1$, $i = 0, 1, \cdots, d-1$, and $b_d = 1$ (see Figure A1). The ith stage is a generalized exponential server of mean rate μ_i, where μ_i may be complex. Define $A_\phi = a_0 a_1 \cdots a_{\phi-1}$, $1 \leq \phi \leq d$. The mean service time μ is then given by $1/\mu = \sum_{l=1}^{d} A_l \mu_l$.

Now assume a d_c-stage Cox representation for the LST, $\Phi_c(s)$, of class c customers. Consider the state process $\mathbf{z}(t) = (z_{c,l}(t) : c \in \mathcal{C},\ 1 \leq l \leq d_c)$, where $z_{c,l}(t)$ denotes the number of class c customers in the lth stage of service. Define the set of feasible states as $\mathcal{F} = \{\mathbf{z} : \sum_{c \in \mathcal{C}} \sum_{l=1}^{d_c} A_c z_{c,l} \leq B\}$ and the *blocking states* for $c \in \mathcal{C}$ as $\mathcal{F}_c(B) = \{\mathbf{z} : \sum_{c \in \mathcal{C}} \sum_{l=1}^{d_c} A_c z_{c,l} > B - A_c\}$. Let $P_{\mathbf{Z}}(\mathbf{z})$ denote the equilibrium distribution of $\mathbf{z}(t)$. $\mathbf{z}(t)$ is a Markov process which does not change when an arriving customer is blocked. Therefore, we introduce a *flip-flop* process $f(t) \in \{0, 1\}$, which changes value each time a blocking event occurs. The joint process $\mathbf{v}(t) = (\mathbf{z}(t), f(t))$ is Markov, with equilibrium distribution $P_{\mathbf{Z},f}(\mathbf{z}, f)$.

The proof proceeds by conjecturing the form of the distribution $P_{\mathbf{Z},f}(\mathbf{z}, f)$ and the reverse process $\mathbf{v}_R(t) = \mathbf{v}(-t)$. In particular, we conjecture that

$$P_{\mathbf{Z},f}(\mathbf{z}, f) = \frac{1}{2} P_{\mathbf{Z}}(\mathbf{z}), \quad f \in \{0, 1\} \quad (29)$$

where

$$P_{\mathbf{Z}}(z) = P_{\mathbf{Z}}(0) n! \prod_{c \in \mathcal{C}} \prod_{l=1}^{d_c} \frac{\rho_{c,l}^{z_{c,l}}}{z_{c,l}} \quad (30)$$

with $\rho_{c,l} = \lambda_c A_{c,l} / \mu_{c,l}$. The conjectured reverse process $\mathbf{v}_R(t)$ consists of independent Poisson arrivals of rates λ_c, $c \in \mathcal{C}$, with the station operating as in forward time, but with the holding times realized by the Cox representation reversed in time (see Figure A2). In order to establish the truth of these conjectures, it suffices to show that the following *reversed balance equation* (cf. [6]) holds:

$$P_{\mathbf{V}}(\mathbf{v}) q(\mathbf{v}, \mathbf{v}') = P_{\mathbf{V}}(\mathbf{v}') q_R(\mathbf{v}', \mathbf{v}), \quad (31)$$

for all $\mathbf{v}, \mathbf{v}' \in \mathcal{F}(B) \times \{0, 1\}$, where $q(\cdot, \cdot)$ and $q_R(\cdot, \cdot)$ denote the transition rates of the forward and reverse processes $\mathbf{v}(t)$ and $\mathbf{v}_R(t)$, respectively.

There are several transitions of the forward process involving a class c customer, which can occur:

(i) When $\mathbf{z} \in \mathcal{F}(B)$, a class c customer in stage ϕ departs the station, resulting in state $\mathbf{z}'$. Then $q(\mathbf{v}, \mathbf{v}') = z_{c,\phi}\mu_{c,\phi}b_{c,\phi}$ and $q_R(\mathbf{v}', \mathbf{v}) = A_{c,\phi}b_{c,\phi}\lambda_c$, where $\mathbf{v} = (\mathbf{z}, f)$ and $\mathbf{v}' = (\mathbf{z}', f)$.

(ii) When $\mathbf{z} \in \mathcal{F}(B)$, a class c customer in stage $\phi - 1$ moves to stage ϕ, resulting in state $\mathbf{z}'$, for $1 < \phi < d_c$. Then $q(\mathbf{v}, \mathbf{v}') = z_{c,\phi-1}\mu_{c,\phi-1}a_{c,\phi-1}$ and $q_R(\mathbf{v}', \mathbf{v}) = \mu_{c,\phi}(z_{c,\phi} + 1)$. In case $\phi = 1$, an external arrival occurs and we have $q(\mathbf{v}, \mathbf{v}') = \lambda_c a_{c,0}$ and $q_R(\mathbf{v}', \mathbf{v}) = (z_{c,1} + 1)\mu_{c,1}$.

(iii) When $\mathbf{z} \in \mathcal{F}_c(B)$, a class c customer arrives and is blocked. Then $q((\mathbf{z}, f), (\mathbf{z}, 1 - f)) = q_R((\mathbf{z}, 1 - f), (\mathbf{z}, f)) = \lambda_c$.

It is straightforward to verify that these transition rates satisfy (31). This establishes that the Erlang station is quasireversible with equilibrium distribution (29).

By summing $P_{\mathbf{Z}}(\mathbf{z})$ over all states with $\sum_{l=1}^{d_c} z_{c,l} = n_c, \quad c \in \mathcal{C}$, we obtain

$$\pi_B(\mathbf{n}) = G(B)^{-1} \prod_{c \in \mathcal{C}} \frac{a_c^{n_c}}{n_c!}, \quad \mathbf{n} \in \mathcal{S}(B) \qquad (32)$$

where $n_c = \sum_{l=1}^{d_c} a_{c,l} = \lambda_c/\mu_c$. To establish the reversibility of $\mathbf{n}(t)$, it suffices to check that $\pi_B(\mathbf{n})$ satisfies the detailed balance equations:

$$\lambda_c \pi_B(\mathbf{n}_c^-) = n_c \mu_c \pi_B(\mathbf{n}), \quad \mathbf{n}, \mathbf{n}_c^- \in \mathcal{S}(B), \qquad (33)$$

where $\mathbf{n}_c^-$ denotes the state vector with one less class c customer than $\mathbf{n}$. □

Acknowledgements

The authors would like to thank Prof. Yaakov Kogan of Technion and Dr. Alex Birman of IBM Research for stimulating discussions on the topic of loss networks.

References

[1] F. Baskett, K.M. Chandy, R.R. Muntz, and F.C. Palacios, "Open, closed, and mixed networks of queues with different classes of customers," *Journal ACM*, **22**(2), pp. 248-260, 1975.

[2] J.W. Cohen, "The Generalized Engset Formulae," *Philips Telecommunication Review*, **18**(4):158-170, November 1957.

[3] Z. Dziong and J.W. Roberts, "Congestion probabilities in a circuit-switched integrated services network," *Performance Evaluation*, Vol. 7, pp. 267-284, 1987.

[4] J.R. Jackson, "Jobshop-like Queueing Systems," *Management Science*, **10**(1) : 131 - 142, October 1963.

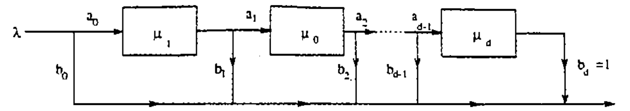

Figure A1: Cox Representation of General Holding Time

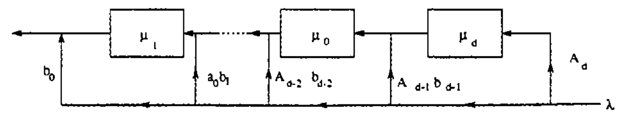

Figure A2: Cox Representation in Reverse Time

[5] F.P. Kelly, "Loss networks," *Annals of Applied Probability*, Vol. 1, No. 3, pp. 319-378, 1991.

[6] F.P. Kelly, *Reversibility and Stochastic Networks*, Wiley, Chichester, 1979.

[7] H. Kobayashi, *Modeling and Analysis: An Introduction to System Performance Evaluation Methodology*, Addison-Wesley, 1978.

[8] H. Kobayashi, A. Birman, and Y. Kogan, "Algorithmic Analysis and Congestion Control of Connection-Oriented Services in Large Scale Communication Networks," A joint research document, November 1993.

[9] Y. Kogan, "Asymptotic Solution of Generalized Multiclass Engset Model," submitted for publication, 1993.

[10] E. Pinsky and A. Conway, "Exact computation of blocking probabilities in multi-facility blocking models," in *Proc. IFIP WG 7.3 Int. Conf. on the Performance of Distributed Systems and Integrated Communication Networks*, Eds., T. Hasegawa, H. Takagi, Y. Takahashi, Kyoto, Japan, 1991.

[11] M. Reiser and H. Kobayashi, "Queueing networks with multiple closed chains: theory and computational algorithms," *IBM Journal of Research and Development*, **19**: 283-294, 1975.

[12] R. Syski, *Introduction to Congestion Theory in Telephone Systems*. Second Edition, Elsevier Science Publisher, Amsterdam, 1986.

[13] W. Whitt, "Blocking when service is required from several facilities simultaneously," *AT&T Technical Journal*, 64, pp. 1807-1856, 1985.

[14] R. Wolff, *Stochastic Modeling and the Theory of Queues*, Prentice-Hall, Englewood Cliffs, New Jersey, 1989.