# Generalized Multihop Shuffle Networks

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*Abstract*—In this paper, we provide an in-depth study of the generalization of the well-known shuffle network for ultrafast multihop lightwave communication. In the classical definition of a shuffle network, i.e., $N = kp^k$ where $N$ is the number of nodes and $k$ is the number of stages with nodes of degree $p$, the realizable values of $N$ are very sparse and many of the intermediate values of $N$ are not realizable. We use a new definition of the shuffle network, $N = nk$, where $n$ is the number of nodes per stage, which was originally proposed as the shuffle-ring network in [7]. Based on this definition, we divide the shuffle networks into two classes: extra-stage and reduced-stage. We derive an exact model and an approximate model of the expected number of hops for various network topologies. The results can be used to determine an optimal network topology when given a value of $N$.

*Index Terms*—Multihop networks, shuffle networks, generalized shuffle networks, expected number of hops, expected number of nodes

## I. INTRODUCTION

**M**ULTIHOP networks with wavelength-division multiplexing (WDM) are one possible way to conduct high data-rate communication (over 100 Gb/s) [1]. The nodes in multihop networks are connected by optical fiber which has a tremendous bandwidth of tens of terabits per second. Regular two-connected network topologies, such as the Manhattan street network (MSN) [2] and the shuffle network [3], have been considered for possible multihop networks. Specifically, the shuffle network, which has a multistage topology, has been investigated by many researchers. An $N(= kp^k)$ node shuffle network is characterized by two parameters $p$ and $k$, and is represented as a $(p, k)$ shuffle network, where $p$ is the node degree of the network and $k$ is the number of columns. In a $(p, k)$ shuffle network, $p^k$ nodes are linearly arranged in a column, and two adjacent columns are connected in a perfect shuffle by unidirectional links. The last column is wrapped around to the first column in a cylindrical manner. Packets can recirculate through the network until they arrive at their destinations. Having all of the nodes arranged in a single column is known as a single-stage shuffle exchange network. The shuffle network is a regular network, in that each node

has the same number of input and output links. The shuffle network is also an isotropic network, which means that every node sees the network in an identical way, and the spanning tree rooted at each node is the same for all nodes.

One of the most important factors determining the overall throughput of a multihop network is the packet routing control (algorithm). Deflection routing (also called hot-potato [4] routing with no buffer) is a technique that maintains required buffer size to be bounded, while providing a compatible network performance to the store-and-forward scheme [5], [6]. If deflection routing is used as a control principle, network throughput may be contingent upon network topology. Many research results have been reported on the effect of network topology variations on the performance of deflection routing [7], [8]. In this paper, we examine another way to improve network performance through topology variations. A new definition of a shuffle network is used, which differs from the conventional one. This new definition was originally proposed by Krishna and Hajek in analyzing the performance of shuffle-ring type networks [7]. In their new definition, the tight relationship (i.e., $N = kp^k$ for given $N$) between the number of stages $(k)$ and the number of nodes per stage $(n)$ is removed. The network topology becomes more flexible by allowing two independent variables whose product is $N$, i.e., $N = kn$. The new definition enables various distinct values of $N$ to be realized into a shuffle network. We provide an in-depth study of the performance of new shuffle networks by showing that different network topologies provide us with different network performance; by selecting a network topology properly, performance may be improved considerably. We concentrate mainly on the analysis and derivation of the expected number of hops. Throughout this paper, we consider shuffle networks with $p = 2$ for the comparison of various networks, while we derive general formulas for arbitrary $p$.

The remainder of the paper is organized as follows. In Section II, we define a generalized shuffle network by comparing it with a classical shuffle network. The performance analysis of various network topologies is followed in Section III. In that section, we derive an accurate model and an approximate model of the expected number of hops for different cases. Results are summarized and remarks are presented in Section IV.

## II. GENERALIZED SHUFFLE NETWORKS

One important property of a classical shuffle network is its regular structure. However, this requirement always fixes a shuffle network as having $k$ stages with $p^k$ nodes at each stage. In this classical definition of a shuffle network, feasible

numbers of nodes are so sparse that many values of $N$ cannot be realized. For example, assuming $p = 2$, if $k = 1$ then $N = 2$, if $k = 2$ then $N = 8$, if $k = 3$ then $N = 24$ and so on. As can be seen, these numbers are very discrete; thus, other given numbers, e.g., $N = 16, 32, \cdots$, may not be realized into a shuffle network.

To solve this problem, we use a generalization of the shuffle network's structure [7]. In the new definition, they eliminate the tight relationship between $k$ and $n$, instead allowing them to behave as two independent variables whose product equals $N$. A generalized shuffle network is defined as follows.

*Definition 2.1:* A generalized shuffle network (GSN) is constructed with $N = kn$ nodes where $k$ is the number of stages and $n$ is the number of nodes per stage. Here, $n$ is the power of $p$. At each stage, $n$ nodes are linearly arranged, and two adjacent columns are connected in a perfect shuffle by unidirectional links.

The stages in the GSN are numbered as $1, 2, \cdots, k$ from the leftmost stage to the rightmost stage. Each node at stage $i, 1 \leq i \leq k-1$, has $p$ links directed to $p$ nodes at the next stage $i+1$. If the nodes are numbered from zero to $n-1$ at each stage, a node $j$ at stage $i$ is connected to nodes, $j', j'+1, \cdots, j'+p-1$ at stage $i+1$, where $j' = (j \mod p^{k-1})p$, in a perfect shuffle [3]. As in the classical shuffle network, the last stage of a GSN is connected to the first stage in a wrapped-around manner. While a GSN is characterized by four parameters $(N, k, n, p)$, since we are only investigating the relationship between $k$ and $n$ for a given $p$, from now on we denote a GSN simply as $(N, k, n)$.

### III. PERFORMANCE ANALYSIS OF THE GSN

In this section, we analyze the performance of the GSN. Particularly, we focus on deriving the expected number of hops in various network topologies. Since a GSN is characterized basically by two parameters $k$ and $n$, we will determine the relationship between these two parameters. Eventually, we will ascertain the optimal values of $k$ and $n$ to achieve the smallest value of expected hops in the network. Note that in the remainder of the paper we do not assume a specific control algorithm, but the results can be used with any algorithm including store-and-forward and deflection routing.

Let the variable $k'$ denote the number of stages in a conventional shuffle network with $n(= p^{k'})$ nodes per stage, i.e., $k' = \log_p n$. However, note that in a GSN, $k'$ is a virtual number of stages, while $k$ is an actual number of stages. In the following analysis, we consider two cases separately, i.e., $k \geq k'$ and $k < k'$. If $k \geq k'$, we call the networks extra-stage shuffle networks, and if $k < k'$, we call them reduced-stage shuffle networks. If $k = k'$, it is the conventional shuffle network, and is a special case of the extra-stage shuffle networks. If a node is met more than once along a path from source to destination, it is overlapping. In both GSN cases, a source node begins to access every node in a stage after $k'$ hops, if overlapping is not considered. We call this point a saturation point. After the saturation point in extra-stage shuffle networks, the source node encounters the same number of intermediate nodes $(n)$ at each stage at each hop, until it
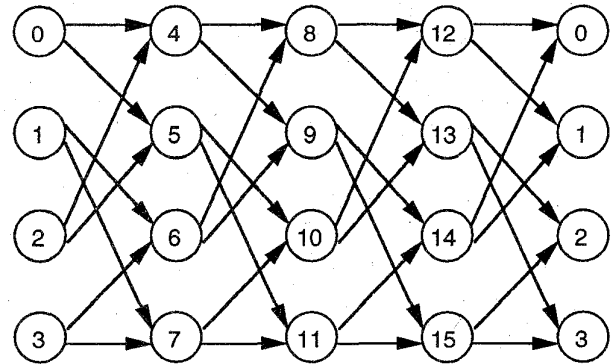


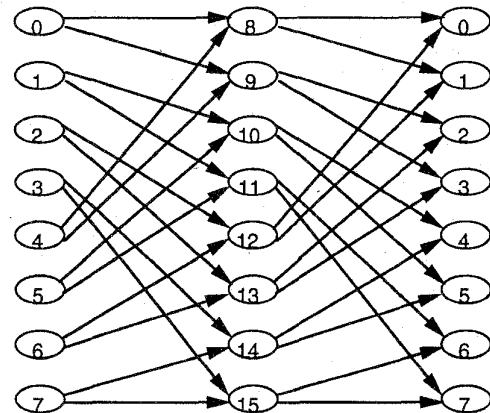Fig. 1.   (16, 4, 4) extra-stage shuffle network for $p = 2$.



Fig. 2.   (16, 2, 8) reduced-stage shuffle network for $p = 2$.

meets previously traversed nodes. However, in reduced-stage shuffle networks, the overlapping begins to appear before the saturation point, which makes the exact analysis of reduced-stage shuffle networks very difficult. The extra-stage shuffle network (16, 4, 4) is shown in Fig. 1 for $p = 2$. Fig. 2 shows an example of the reduced-stage shuffle networks, (16, 2, 8) for $p = 2$.

#### A. Case $k \geq k'$: Extra-stage Shuffle Networks

In extra-stage shuffle networks, network performance parameters are easily calculated due to structural regularity. The number of nodes reachable after $h$ hops from a source is given in Table I. It is easy to see that $k'$ is the saturation point; after $k'$ hops, the number of accessible nodes is fixed to $n$, and decreases after $k$ hops. As a result, the expected number of hops is given by

$$E\{\text{hops}\} = \frac{1}{kn - 1}\left[\sum_{j=1}^{k'-1} jp^j + n \sum_{j=0}^{k-k'-1} (j + k') + \sum_{j=0}^{k'-1} (j + k)(n - p^j)\right].$$

TABLE I
NUMBER OF NODES REACHABLE AT EACH HOP FOR $k \geq k'$

| $h$ | # of nodes |
|---|---|
| 1 | $p$ |
| 2 | $p^2$ |
| $\vdots$ | $\vdots$ |
| $k' - 1$ | $p^{k'-1}$ |
| $k'$ | $p^{k'} = n$ |
| $k' + 1$ | $n$ |
| $\vdots$ | $\vdots$ |
| $k - 1$ | $n$ |
| $k$ | $n - 1$ |
| $k + 1$ | $n - p$ |
| $\vdots$ | $\vdots$ |
| $k + k' - 1$ | $n - p^{k'-1}$ |

The summations are simply calculated in a closed form by substituting $k'$ with $\log_p n$

$$E\{\text{hops}\} = \frac{1}{kn - 1}\left[\frac{n}{2}k^2 - k\frac{n-1}{p-1} - \frac{nk}{2} + nk\log_p n\right]. \tag{1}$$

By substituting $kn$ with $N$ and by eliminating $k$, the expected number of hops can be represented as

$$E\{\text{hops}\} = \frac{1}{N - 1}\left[\frac{N^2}{2n} - \frac{N(n-1)}{n(p-1)} - \frac{N}{2} + N\log_p n\right]. \tag{2}$$

Taking the derivative of $E\{\text{hops}\}$, the optimal value of $n$ to make the function minimal can be obtained

$$\frac{d}{dn}E\{\text{hops}\} = \frac{1}{N - 1}\left[-\frac{N^2}{2n^2} - \frac{N}{(p-1)n^2} + \frac{N}{n}\log_p e\right]. \tag{3}$$

Letting $(d/dn)E\{\text{hops}\} = 0$, the minimal value of $n$ for $k \geq k'$ can be given as

$$n = \left\langle\left(\frac{1}{p-1} + \frac{N}{2}\right)\ln p\right\rangle \tag{4}$$

where $\langle x\rangle$ means the nearest integer of power of two to $x$. Note that $n$ should always be selected to satisfy $k \geq k'$; in other words, $n\log_p n \leq N$.

Furthermore, especially if $n = p^k$, the above equation can be simplified as follows:

$$E\{\text{hops}\} = \frac{1}{2(p-1)(N-1)} \cdot [N(3k-1)(p-1) - 2k(p^k - 1)] \tag{5}$$

which is the same result as in [3]. In Table III, some calculations of the expected number of hops for different $n$'s in case $k \geq k'$ are shown ($p = 2$). The results in Table III indicate that as the number of stages increases, the expected number of hops also increases. This phenomenon matches a simple observation of networks, because more hops are needed as the number of stages increases, due to the saturation effect of the network. As a guideline in selecting a network topology for $k \geq k'$, it is always preferable to keep $k$ as small as possible.
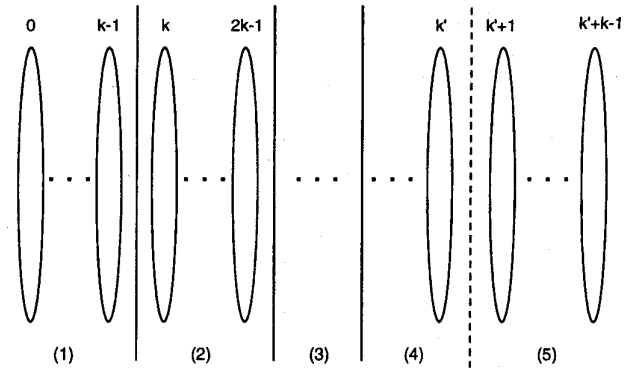


Fig. 3. The five parts traversed by a packet—each part is separated by a solid line from the adjacent part based on the hops to make. The dotted line represents a saturation point: (1) $1 \leq h \leq k - 1$, (2) $k \leq h \leq 2k - 1$, (3) $2k \leq h \leq \lfloor k'/k\rfloor k - 1$, (4) $\lfloor k'/k\rfloor k \leq h \leq k'$, and (5) $k' + 1 \leq h \leq k' + k - 1$.

## B. In Case $k < k'$: Reduced-Stage Shuffle Networks

In a reduced-stage network case, each of the source nodes meets a different number of intermediate nodes at each hop as it progresses to its destination. This means reduced-stage networks are not isotropic, and the spanning trees at each node are not identical. Since $k < k'$, some nodes (stages) may be visited more than once until a packet arrives at the saturation point ($k'$). This phenomenon is illustrated in Fig. 3. Generally, a packet may go through five different parts, i.e., $1 \leq h \leq k - 1, k \leq h \leq 2k - 1, 2k \leq h \leq \lfloor k'/k\rfloor k - 1, \lfloor k'/k\rfloor k \leq h \leq k'$ and $k' + 1 \leq h \leq k' + k - 1$. In $k \leq h \leq 2k - 1$, overlapping between stages begins to appear and continues until the destination stage. Since every source node encounters a different number of intermediate nodes along the way to its destination, one possible method is to obtain the expected number of nodes at each hop. However, even this becomes very complex if the network size increases.

The relationship between overlapped stages can be better seen in an example for $k = 2$ and $p = 2$, as shown in Fig. 4. In Fig. 4, possible paths from a source node to a destination node are represented by a tree. This tree is binary, and the source node is a root node. The destination node resides at one of the leaf nodes. Since $k = 2$, the source node (the leftmost black node in Fig. 4) may be visited again at the second hop in the worst case. If this happens, the gray-colored nodes are automatically revisited at the third hop. At the fourth hop, the overlapping effect becomes more complicated because the nodes visited at the first and third hops appear at the same time in an interfered manner. Hence, to make the analysis tractable, we derive an approximate model for the expected number of hops for $h \geq 2k$, while we derive an exact model up to $h = 2k - 1$.

*1) Expected Number of Hops for $k < k'$:* The expected number of nodes reachable at each hop is listed in Table II. The first part, $1 \leq h \leq k - 1$, in Table II is trivial because there is no overlapping until $h = k$; at the $h$th hop, the packet meets $p^h$ nodes. The second part, $k \leq h \leq 2k - 1$, reflects the overlapping phenomenon between the stages, $1 \leq h \leq k - 1$ and $k \leq h \leq 2k - 1$. At the $k$th stage,

TABLE II
EXPECTED NUMBER OF NODES REACHABLE AT EACH HOP FOR $k < k'$

| $h$ | # of nodes |
|---|---|
| 1 | $p$ |
| 2 | $p^2$ |
| $\vdots$ | $\vdots$ |
| $k-1$ | $p^{k-1}$ |
| $k$ | $(p^k - 1)\frac{k}{N}p^k + p^k\frac{n-p^k}{N}k$ |
| $k+1$ | $(p^{k+1} - p)\frac{2k}{N}p^k + p^{k+1}\frac{(n-2p^k)}{N}k$ |
| $\vdots$ | $\vdots$ |
| $2k-1$ | $(p^{2k-1} - p^{k-1})\frac{k^2}{N}p^k + p^{2k-1}\frac{(n-kp^k)}{N}k$ |
| $2k$ | $\{(p^{2k} - p^k) + (p^{2k} - p^k) + (p^{2k} - p^0) + p^{2k}\}\frac{1}{4} = (4p^{2k} - 2p^k - 2p^0)\frac{1}{4}$ |
| $2k+1$ | $(4p^{2k+1} - 2p^{k+1} - 2p^1)\frac{1}{4}$ |
| $\vdots$ | $\vdots$ |
| $3k-1$ | $(4p^{3k-1} - 2p^{2k-1} - 2p^{k-1})\frac{1}{4}$ |
| $3k$ | $(8p^{3k} - 4p^{2k} - 4p^k - 4p^0)\frac{1}{8}$ |
| $3k+1$ | $(8p^{3k+1} - 4p^{2k+1} - 4p^{k+1} - 4p^1)\frac{1}{8}$ |
| $\vdots$ | $\vdots$ |
| $4k-1$ | $(8p^{4k-1} - 4p^{3k-1} - 4p^{2k-1} - 4p^{k-1})\frac{1}{8}$ |
| $\vdots$ | $\vdots$ |
| $(\lfloor\frac{k'}{k}\rfloor - 1)k$ | $p^{(\lfloor\frac{k'}{k}\rfloor-1)k} - \frac{1}{2}\sum_{h=0}^{\lfloor\frac{k'}{k}\rfloor-2} p^{(\lfloor\frac{k'}{k}\rfloor-2)k-kh}$ |
| $\vdots$ | $\vdots$ |
| $\lfloor\frac{k'}{k}\rfloor k - 1$ | $p^{\lfloor\frac{k'}{k}\rfloor k-1} - \frac{1}{2}\sum_{h=0}^{\lfloor\frac{k'}{k}\rfloor-2} p^{\lfloor\frac{k'}{k}\rfloor k-kh-k-1}$ |
| $\lfloor\frac{k'}{k}\rfloor k$ | $p^{\lfloor\frac{k'}{k}\rfloor k} - \frac{1}{2}\sum_{h=0}^{\lfloor\frac{k'}{k}\rfloor-1} p^{(\lfloor\frac{k'}{k}\rfloor-1)k-kh}$ |
| $\vdots$ | $\vdots$ |
| $k'$ | $p^{k'} - \frac{1}{2}\sum_{h=0}^{\lfloor\frac{k'}{k}\rfloor-1} p^{k'-k-kh}$ |
| $k'+1$ | $n - \frac{1}{2}\sum_{h=0}^{\lfloor\frac{k'}{k}\rfloor-1} p^{\lfloor\frac{k'}{k}\rfloor k-kh}$ |
| $\vdots$ | $\vdots$ |
| $k'+k-1$ | $n - \frac{1}{2}\sum_{h=0}^{\lfloor\frac{k'}{k}\rfloor-1} p^{\lfloor\frac{k'}{k}\rfloor k-kh+k-2}$ |

one is a previously visited node at the first stage as a source node, while $p^k - 1$ nodes are newly visited. Hence, we can derive an expected number of nodes to visit at the $k$th hop as $(p^k - 1)(p^k/N)k + p^k(n - p^k/N)k$ for all $N$ nodes. More specifically, among $n$ source nodes at the first stage, $p^k$ nodes meet $(p^k - p^0)$ nodes at the $(k+1)$th stage after $k$ hops, and $(n - p^k)$ nodes at the first stage meet $p^k$ nodes at the $(k+1)$th stage in a regular manner.

In $2k \le h \le 3k - 1$, overlapping becomes more complex and the previous two parts, $1 \le h \le k-1$ and $k \le h \le 2k-1$, have a simultaneous effect on $2k \le h \le 3k-1$. When $h = 2k$, the source node at the first stage may meet itself again, as it may when $h = k$. A packet meets a different number of nodes at $h = 2k$ depending on the source node. In other words, a packet from a certain source node may meet $I_1 = p^{2k}$ nodes, and other packets from different source nodes may encounter $I_2 = p^{2k} - p^0, I_3 = p^{2k} - p^k, I_4 = p^{2k} - p^k - p^0$ nodes.

respectively (see Fig. 4). Let $\rho_i$ be a probability that a packet will meet $I_i$ nodes at $h = 2k$ hops. Then, the expected number of nodes at $h = 2k$ hops can be computed as

$$E\{\text{nodes}\}_{h=2k} = \sum_{i=1}^{4} I_i\rho_i$$

where the probability $\rho_i$ is given as seen in the equation at the bottom of the page. In general, at the $jk$th hop, $2k \le jk \le \lfloor k'/k\rfloor k - 1$, a packet may meet $\Gamma = \Sigma_{r=1}^{j} \binom{j}{r}$ different sets of intermediate nodes depending on the source node position. Similarly, $\rho_i$ can be defined as a probability that a packet meets $I_i$ nodes at $h = jk$ hops where $1 \le i \le \Gamma$.

The preciseness of modeling implies all of the probabilities should be calculated accurately in order to obtain the expected number of hops. However, as $\lfloor k'/k\rfloor$ increases, obtaining the exact value of $\rho_i$ becomes more and more complex because as a packet propagates further to the following parts, the previous

$$\rho_i = \frac{\text{number of source nodes from which a packet meets } I_i \text{ nodes at } h = 2k \text{ hops}}{N}.$$
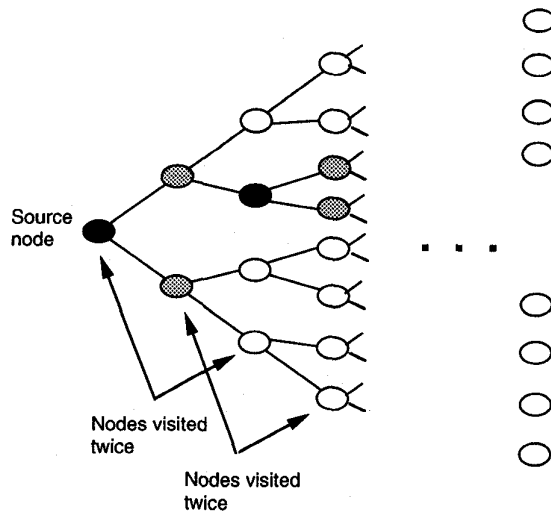
Fig. 4.   Tree representation of possible paths for $k = 2$ and $p = 2$.

parts affect the present part simultaneously. (Furthermore, for example, the third part, $2k \leq h \leq \lfloor k'/k \rfloor - 1$, may be composed of multiple stages of $k$.) One possible way to an approximate modeling is to assume that all of the probabilities are equal, i.e., $\rho_i = \rho$. Table II shows an approximate model, assuming equal probabilities in $\rho_i$. For example, at $2k \leq h \leq 3k-1$, $\rho_i$'s are set to be $\frac{1}{4}$. Similarly, at $3k \leq h \leq 4k-1$, $\rho_i$'s are set to be $\frac{1}{8}$, and so on.

At $\lfloor k'/k \rfloor k \leq h \leq k'$, a similar approximation is applied up to $h = k'$, because the network begins to saturate at $h = k'+1$. If a network reaches a saturation point (the last part of Table II), the number nodes accessible at each hop cannot exceed $n$. Based on Table II, the equation for the expected number of hops for reduced-stage shuffle networks, multiplied by $(kn - 1)$, is derived as follows:

$$
\begin{aligned}
(kn - 1)E\{\text{hops}\} \\
= \sum_{j=1}^{k-1} jp^j + \sum_{j=0}^{k-1}(k+j)\left[(p^{k+j} - p^j)p^k\frac{(j+1)k}{N}\right. \\
\left. + kp^{k+j}\frac{n - (j+1)p^k}{N}\right] \\
+ \sum_{i=2}^{\lfloor k'/k \rfloor-1}\sum_{j=0}^{k-1}(ik + j)\left[p^{ik+j} - \frac{1}{2}\sum_{h=0}^{i-1}p^{(i-1)k-hk+j}\right] \\
+ \sum_{j=0}^{k'-\lfloor k'/k \rfloor k}\left(\left\lfloor\frac{k'}{k}\right\rfloor k + j\right)\left[p^{\lfloor k'/k \rfloor k+j}\right. \\
\left. - \frac{1}{2}\sum_{h=0}^{\lfloor k'/k \rfloor-1}p^{(\lfloor k'/k \rfloor-1)k-kh+j}\right] \\
+ \sum_{j=0}^{k-2}(k' + 1 + j)\left[n - \frac{1}{2}\sum_{h=0}^{\lfloor k'/k \rfloor-1}p^{\lfloor k'/k \rfloor k-kh+j}\right].
\end{aligned}
$$

$$(6)$$

Letting $\alpha = \lfloor k'/k \rfloor$ where $k' = \alpha k + \beta$, this equation can be written into a closed form as follows:

$$
\begin{aligned}
(kn - 1)E\{\text{hops}\} \\
= B(k) + kp^k\left(1 - \frac{1}{n}\right)A(k) + p^k\left(1 - \frac{1}{n} - \frac{k}{n}\right) \\
\cdot B(k) - \frac{p^k}{n}C(k) \\
+ A(k)\{E(\alpha) - kp^k\} + \{D(\alpha) - p^k - 1\}B(k) \\
- \frac{A(k)\{E(\alpha) - kp^k\}}{2(p^k - 1)} \\
+ \frac{kA(k)}{2(p^k - 1)}\left[\frac{1}{2}\alpha(\alpha - 1) - 1\right] \\
- \frac{\{D(\alpha) - p^k - 1\}B(k)}{2(p^k - 1)} + \frac{B(k)}{2(p^k - 1)}(\alpha - 2) \\
+ \{\alpha kA(\beta) + B(\beta)\}\left[p^{\alpha k} - \frac{1}{2}p^{(\alpha-1)k}F(\alpha)\right] \\
+ (k' + 1)(k - 1)n + \frac{n(k - 1)(k - 2)}{2} \\
- \frac{1}{2}p^{\alpha k}F(k - 1)\{(k' + 1)A(k - 1) \\
+ B(k - 1) - (k - 1)p^{k-1}\}
\end{aligned}
$$

$$(7)$$

where $A(x), B(x), C(x), D(x), E(x),$ and $F(x)$ are defined as

$$A(x) = \sum_{j=0}^{x-1} p^j = \frac{p^x - 1}{p - 1}$$

$$B(x) = \sum_{j=0}^{x-1} jp^j = \frac{xp^x}{p - 1} - \frac{p(p^x - 1)}{(p - 1)^2}$$

$$
\begin{aligned}
C(x) = \sum_{j=0}^{x-1} j^2p^j &= \frac{x^2p^x(p - 1) - xp^{x+1}}{(p - 1)^2} \\
&- \frac{\{(x + 1)p^{x+1} - p\}(p - 1) - 2p^2(p^x - 1)}{(p - 1)^3}
\end{aligned}
$$

$$D(x) = \sum_{j=0}^{x-1} p^{jk} = \frac{p^{kx} - 1}{p^k - 1}$$

$$E(x) = \sum_{j=0}^{x-1} jkp^{jk} = \frac{kxp^{kx}}{p^k - 1} - \frac{kp^k(p^{kx} - 1)}{(p^k - 1)^2}$$

$$F(x) = \sum_{j=0}^{x-1} p^{-kj} = \frac{p^{-kx} - 1}{p^{-k} - 1}.$$

Especially if $k < k' < 2k$, the saturation occurs in $k < h < 2k$, and destination nodes lie in $2k \leq h \leq 3k - 1$. In this case, a much simpler expression can be derived from Table II. The expected number of hops, multiplied by $(kn - 1)$, is represented as

$$
\begin{aligned}
(kn - 1)E\{\text{hops}\} \\
= \sum_{j=1}^{k-1} jp^j + \sum_{j=0}^{k'-k}(k+j)\left[(p^{k+j} - p^j)\right. \\
\left. \cdot p^k\frac{(j+1)k}{N} + kp^{k+j}\frac{n - (j+1)p^k}{N}\right]
\end{aligned}
$$

TABLE III
EXPECTED NUMBER OF HOPS FOR DIFFERENT $n$'s

| | $n=2$ | 4 | 8 | 16 | 32 | 64 | 128 | 256 | 512 |
|---|---|---|---|---|---|---|---|---|---|
| $N=8$ | 2.286 | **2.000** | 2.107* | × | × | × | × | × | × |
| 16 | 4.267 | 2.933 | 2.730* | 2.833* | × | × | × | × | × |
| 24 | 6.261 | 3.913 | **3.261** | × | × | × | × | × | × |
| 32 | 8.258 | 4.903 | 3.742 | 3.565* | 3.693* | × | × | × | × |
| 64 | 16.254 | 8.889 | 5.714 | **4.635** | 4.511* | 4.595* | × | × | × |
| 128 | 32.252 | 16.882 | 9.701 | 6.614 | 5.635* | 5.380* | 5.512* | × | × |
| 160 | 40.252 | 20.881 | 11.698 | 7.610 | **6.069** | × | × | × | × |
| 256 | 64.251 | 32.878 | 17.694 | 10.604 | 7.561 | 6.923* | 6.708* | 6.834* | × |
| 384 | 96.251 | 48.877 | 25.691 | 14.601 | 9.556 | **7.536** | 7.892* | × | × |
| 512 | 128.250 | 64.877 | 33.691 | 18.599 | 11.554 | 8.532 | 8.031* | 7.648* | 7.965* |

†Numbers marked with * correspond to reduced stage shuffle networks.
‡Numbers in boldface correspond to classical shuffle networks.
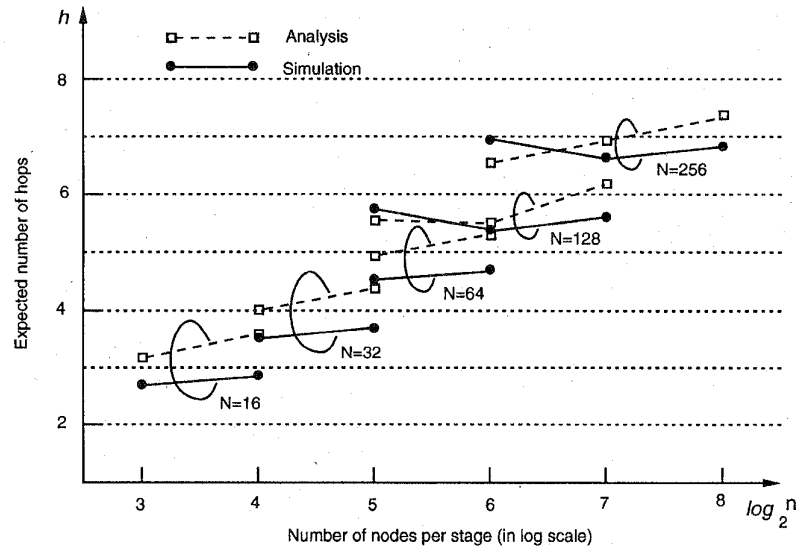‡†Each square marked with × is not feasible.



Fig. 5. Comparison between analysis and simulation results in reduced-stage shuffle networks.

$$+ \sum_{j=1}^{2k-k'-1} (k'+j)\left[(n-p^j)p^k \frac{(k'-k+1)k}{N}\right.$$

$$\left. + kp^{k+j} \frac{n-(k'-k+1)p^k}{N}\right]$$

$$+ \sum_{j=0}^{k'-k-1} (2k+j)(n-p^{k+j}-p^j). \qquad (8)$$

*2) Discussions:* Table III shows some of the results of the expected number of hops for different $n$'s. The network topology for $k = 1$ corresponds to a single-stage shuffle network. Note that the expected number of hops in a single-stage network is not minimal compared to multiple stage cases, even though a single-stage network has a minimum diameter in a class of shuffle networks. This phenomenon originates from the fact that there are many node overlappings in a single-stage network, which results in a low efficiency. The efficiency of a single-stage shuffle network can be improved

by considering different connections between some nodes to avoid self-loops which causes a lot of node overlapping. For example, Maxemchuk considered an alternative architecture of single-stage networks in [5] where node 0 connects to nodes 1 and $(N-1)$, and node $(N-1)$ connects to nodes 0 and $(N-2)$.

To substantiate the validity of the equal probability assumption in reduced-stage shuffle networks, simulation was conducted by measuring the actual number of hops taken by a test packet for various destinations. Fig. 5 shows that the approximate model, based on the equal probability assumption, predicts the expected number of hops fairly well, and matches simulation results.

## IV. CONCLUSION

In this paper, we analyzed different topologies of shuffle networks. Unlike the classical definition of a shuffle network, i.e., $N = kp^k$, we consider a generalization of the definition,

i.e., $N = kn$. This definition makes it possible to realize a shuffle network in a variety of different ways with a given $N$. Different network topologies may provide us with different network throughputs; and by selecting a network properly, significant performance improvement can be achieved. The reduced-stage shuffle networks have a shorter network diameter and offer smaller expected number of hops, but experience a lot of node overlapping in a certain source–destination path as $k$ becomes smaller. On the other hand, the performance of extra-stage shuffle networks becomes worse as $k$ becomes larger due to the long diameter. However, extra-stage shuffle networks offer a larger fraction of don't care nodes, which is desirable for deflection routing. Study results can be used to determine an optimal network topology when given a value of $N$, by trading off the complexity and performance of a network.

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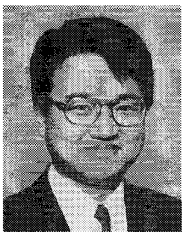
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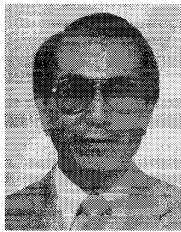
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