Performance Analysis of Adaptive Fixed Assignment Protocol

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Abstract

In this paper we conduct a performance analysis for the adaptive fixed assignment protocol (AFAP) presented in [8]. AFAP is a simple mixture of interleaved TDMA (I-TDMA*) and interleaved slotted ALOHA (I-SA) protocols [1]. However, the analysis of AFAP is complicated due to the periodicity of I-TDMA* and the queue interaction in I-SA. We model an arbitrary user-to-channel queue with a periodic Markov chain. The influence of other queues is modelled through the transition probabilities of this chain.

Key Words: Delay Analysis, Slotted ALOHA, Optical Star Network, Periodic Markov Chain.

1 Introduction

In our recent work [7, 8] we proposed a new protocol for a single-hop multichannel optical network, AFAP. Each receiver is tuned to a fixed channel, and channels are assigned to transmitters according to a scheduling scheme. A user declares the use of its time slot through the control channel. Each control packet is C bits long, where C is the number of channels. A channel which is not used by the assigned user will be made available to other users based on an ALOHA type random access scheme. If two or more users contend for the same channel, they are forbidden to access it again until their scheduled time slots arrive. AFAP can achieve both the low latency of ALOHA, and the near 100% efficiency of TDMA.

Several authors [1, 2, 6, 9] analyze slotted ALOHA with a finite number (M) of buffered users. They all resort to an iterative procedure to obtain the system steady state probabilities. In the case of infinite buffers, the system is accurately modelled by the M-dimensional Markov chain whose coordinates take on an infinite number of values. In order to simplify the analysis, Saadawi and Ephermides introduce the system and the user Markov chains. The system Markov chain models the number of active and backlogged users, whereas the user Markov chain models the number of packets in an arbitrary buffer. Transition probabilities of the system Markov chain depend on the steady state probabilities of the user Markov chain and vice versa. Yao and Yang treat each buffer as an M/G/1 queueing system whose service time distribution depends on the steady state probabilities of the system Markov chain. In the case of finite buffers (with size of B cells), the system should be modelled by a Markov chain with B^M states. Bogineni et al. model the I-SA protocol by using only the user Markov chain with B states. The transition probabilities of this chain depend on its own steady state probabilities. We will adopt a similar approach in our analysis.

We model an arbitrary user-to-channel queue with a periodic Markov chain of period M. This queue interacts with queues associated with the same channels due to the collisions. Also, the marked queue interacts with queues associated with the same user whose transmitter may serve just one queue at a time. All queues exhibit the identical statistical behavior which might be shifted in time. Therefore, the transition probabilities of the associated periodic Markov chain depend on its conditional state probabilities as will be explained in more detail later on. Our protocol is exactly modelled by a periodic Markov chain with $M \cdot B^M$ states. However, our approximate model of AFAP uses the periodic Markov chain with $M \cdot B$ states only.

The paper is organized as follows. In section 2 we define AFAP in a manner similar to [8]. In section 3 we develop the periodic Markov chain model and the iterative procedure for calculating the steady state probabilities. Comparison of analytical and simulation results is given in section 4. Section 5 summarizes the paper.

1.1 AFAP Description

We change slightly AFAP in order to simplify the analysis while keeping its performance the same. The AFAP can be defined with the following steps:

- $(k-1)$th time slot: User $(k-j\cdot\frac{M}{C}) \bmod M$ sets the jth bit of the control packet to 1 if it has a packet for the receiver tuned to channel j, not including scheduled packets that are being sent in the $(k-1)$th time slot.
- $(k-1)$th time slot: Users learn which channels will be reserved in the kth time slot. The rest of the channels will be free for access.
- kth time slot: Scheduled users transmit their packets. If the scheduled queue was in the back-off state, it enters the transmit state.
- kth time slot: Other users select randomly free channels to which they will send their packets. If a packet collides, the corresponding queue enters the back-off state. Packets scheduled for the $(k+1)$th time slot are not transmitted in the ALOHA fashion.

Note that there is a change of the fixed scheduling scheme according to which the transmitters tune to the channels. In [8] we adopted the I-TDMA* assignment scheme. Figure 1 gives an example of a new assignment scheme for the case of $M = 8$ users and $C = 4$ channels. Also, each user chooses which free channel to access in a random rather than round-robin fashion. Neither of these changes influences the AFAP performance, but both are simpler to incorporate into our model.

2 Analysis

2.1 Periodic Markov Chain Model

We associate with each queue a Markov chain $\{S_k,\ k \geq 1\}$, where S_k is a triplet (X_k, Y_k, Z_k). Here X_k is the number of packets in the queue at the kth time slot; $Y_k = 1$ if the queue cannot be

USER \ TIME SLOT	0	1	2	3	4	5	6	7
0	0	X	1	X	2	X	3	X
1	X	0	X	1	X	2	X	3
2	3	X	0	X	1	X	2	X
3	X	3	X	0	X	1	X	2
4	2	X	3	X	0	X	1	X
5	X	2	X	3	X	0	X	1
6	1	X	2	X	3	X	0	X
7	X	1	X	2	X	3	X	0

Figure 1: Assignment of channels (entries in the table) during the time slots of a frame in AFAP. The case $M = 8$ and $C = 4$ is shown.

served because of a recent collision, and $Y_k = 0$ otherwise; and Z_k is the elapsed time from the last scheduled time slot. In order to evaluate performance measures such as channel throughput and average packet delay we first need to compute the conditional state probabilities defined by:

$$\pi(i,j|l) = \lim_{k\to\infty} P(X_k = i, Y_k = j | Z_k = l), \qquad (1)$$

where $0 \leq i \leq B$, $0 \leq j \leq 1$, and $0 \leq l \leq M-1$.

Let β denote the probability of a packet arrival; σ, the probability that a scheduled user has a packet to send; $c(l)$, $1 \leq l \leq M-1$, the probability of selecting a given queue for $Z_k = l$; and $\gamma(l)$, $1 \leq l \leq M-1$, the probability of successful transmission for $Z_k = l$. Then the transition probabilities of the Markov chain $\{S_k,\ k \geq 1\}$ are given by the following expressions:

$$\begin{aligned} p_{(0,0,0),(1,0,1)} &= 1 - p_{(0,0,0),(0,0,1)} = \beta, \\ p_{(i,0,0),(i,0,1)} &= 1 - p_{(i,0,0),(i-1,0,1)} = \beta, \\ p_{(B,0,0),(B,0,1)} &= 1 - p_{(B,0,0),(B-1,0,1)} = \beta; \end{aligned}$$

where $1 \leq i \leq B-1$;

$$\begin{aligned} p_{(0,0,l),(1,0,l+1)} &= 1 - p_{(0,0,l),(0,0,l+1)} = \beta, \\ p_{(i,0,l),(i-1,0,l+1)} &= (1-\beta)(1-\sigma)\gamma(l), \\ p_{(i,0,l),(i,0,l+1)} &= (1-\beta)(\sigma + (1-\sigma)(1-c(l))) \\ &\quad + \beta(1-\sigma)\gamma(l), \\ p_{(i,0,l),(i+1,0,l+1)} &= \beta(\sigma + (1-\sigma)(1-c(l))), \end{aligned}$$

$$
\begin{aligned}
p_{(i,0,l),(i,1,l+1)} &= (1-\beta)(1-\sigma)(c(l)-\gamma(l)),\\
p_{(i,0,l),(i+1,1,l+1)} &= \beta(1-\sigma)(c(l)-\gamma(l)),\\
p_{(B,0,l),(B-1,0,l+1)} &= (1-\beta)(1-\sigma)\gamma(l),\\
p_{(B,0,l),(B,0,l+1)} &= \sigma+(1-\sigma)(1-c(l))\\
&\quad +\beta(1-\sigma)\gamma(l),\\
p_{(B,0,l),(B,1,l+1)} &= (1-\sigma)(c(l)-\gamma(l)),
\end{aligned}
$$

where $1 \le l \le M-1$ and $1 \le i \le B-1$;

$$
\begin{aligned}
p_{(i,1,l),(i+1,1,l+1)} &= 1-p_{(i,1,l),(i,1,l+1)}=\beta,\\
p_{(B,1,l),(B,1,l+1)} &= 1,
\end{aligned}
$$

where $2 \le l \le M-2$ and $1 \le i \le B-1$;

$$
\begin{aligned}
p_{(i,1,M-1),(i+1,0,0)} &= 1-p_{(i,1,M-1),(i,0,0)}=\beta,\\
p_{(B,1,M-1),(B,1,0)} &= 1,
\end{aligned}
\tag{2}
$$

where $1 \le i \le B-1$.

Therefore, the transition matrix of this Markov chain has the following structure [5]:

$$
\mathbf{P} = \begin{bmatrix} 0 & \mathbf{P}_{0,1} & 0 & \cdots & 0 \\ 0 & 0 & \mathbf{P}_{1,2} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \mathbf{P}_{(M-2),(M-1)} \\ \mathbf{P}_{(M-1),0} & 0 & 0 & \cdots & 0 \end{bmatrix}, \tag{3}
$$

and:

$$
\mathbf{P}^M = \begin{bmatrix} \mathbf{P}_0 & 0 & \cdots & 0 \\ 0 & \mathbf{P}_1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \mathbf{P}_{M-1} \end{bmatrix}, \tag{4}
$$

where:

$$
\mathbf{P}_l = \mathbf{P}_{l,l+1\,\mathrm{mod}M} \cdots \mathbf{P}_{l-1\,\mathrm{mod}M,l}, \tag{5}
$$

with $0 \le l \le M-1$. The conditional probability vector $\pi_l = \{\pi(i,j|l)|\ 0 \le i \le B,\ 0 \le j \le 1\}$ can be calculated as follows [4]:

$$
\pi_l = \mathbf{1}(\mathbf{E}-\mathbf{I}+\mathbf{P}_l)^{-1}, \tag{6}
$$

where $0 \le l \le M-1$, $\mathbf{1}$ is a column vector of all ones, $\mathbf{I}$ is the identity matrix, and $\mathbf{E}$ is a matrix of all ones.

Unknown parameters in (2), σ, $c(l)$, $\gamma(l)$, $1 \le l \le M-1$, can be expressed in terms of the conditional probabilities defined in (1). The following approximate expressions are derived in the appendix.

The probability that a scheduled queue is not empty is given by

$$
\sigma = 1 - \pi(0,0|0). \tag{7}
$$

The probability that the marked user will decide to transmit a packet over a given channel is approximately given by

$$
c(l) = \frac{1-\sigma \cdot 1_{\{l=0\mathrm{mod}r\}}}{1+(1-\sigma)\sum_{\substack{1\le n\le M-1\\ n=l\mathrm{mod}r\\ n\ne l}} \sum_{i=1}^{B} \pi(i,0|n)}, \tag{8}
$$

where $1 \le l \le M-1$, $r = M/C$ is an integer, and 1_A is the indicator function of event A. Equation (8) incorporates the coupling of the queues associated with the marked user.

The probability of successful transmission is approximated by:

$$
\gamma(l) = c(l) \cdot \prod_{\substack{m=1\\ m\ne l}}^{M-1} \left(1 - c(m) \cdot \sum_{i=1}^{B} \pi(i,0|m)\right), \tag{9}
$$

where $1 \le l \le M-1$. Equation (9) incorporates the effect of collisions in AFAP.

2.2 Iterative Procedure

We find the conditional probabilities in (1) using the iterative procedure:

- Step 1: Initialize σ, $c(l)$, $\gamma(l)$, $1 \le l \le M-1$.
- Step 2: Calculate transition probabilities according to (2).
- Step 3: Calculate conditional state probabilities according to (5) and (6).
- Step 4: Calculate new values for σ, $c(l)$, $\gamma(l)$, $1 \le l \le M-1$, according to (7), (8), and (9). If they do not differ from the previous values by more than a specified ϵ% then stop. Else, go to Step 2.

3 Results

The probability that a packet is sent from a particular queue is given by

$$
S_q = \frac{1}{M}\left(\sigma + (1-\sigma)\sum_{k=1}^{M-1}\gamma(k)\sum_{i=1}^{B}\pi(i,0|k)\right). \tag{10}
$$

The average number of packets in the queue is:

$$N = \frac{1}{M} \sum_{k=0}^{M-1} \sum_{i=1}^{B} i \cdot (\pi(i, 1|k) + \pi(i, 0|k)). \quad (11)$$

The network throughput is simply given by $M \cdot C \cdot S_q$, and the average packet delay is given by N/S_q. In Figure 2, we plot the average packet delay versus network throughput, and confirm a very good agreement of analytical and simulation results.

4 Summary

In this paper we analyze AFAP performance, which was previously assessed only through simulation. Direct Markov chain analysis would involve too large a number of states, and would not be amenable to obtaining a solution even for a small number of users and buffer size. Therefore, we introduce a Markov chain that models an arbitrary queue in the system. Its interection with other queues is modeled only through the Markov chain transition probabilities. Since other queues are represented by the same Markov chain model, their influence on the given queue is a function of its steady state solution. In order to obtain this solution, we resort to an iterative procedure. We compare the analytical and simulation results, demonstrating a good agreement.

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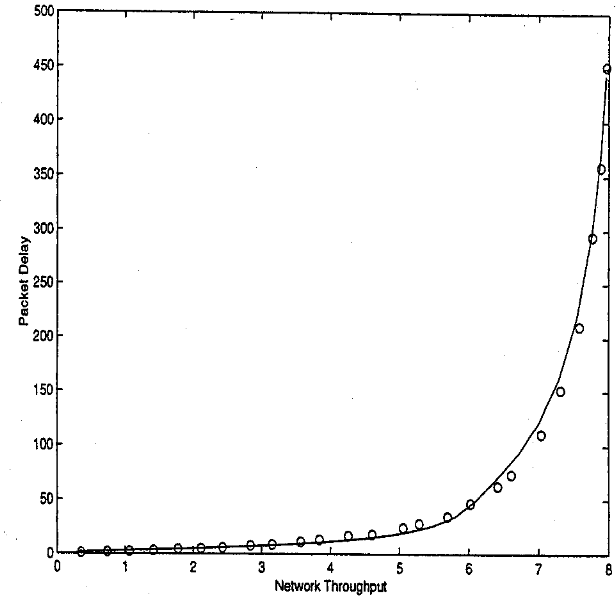
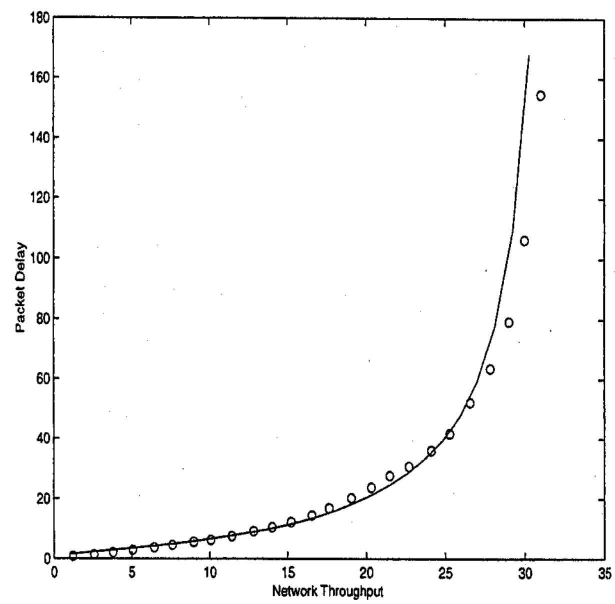
Appendix

Theorem: If $Z_k = l$, $1 \le l \le M-1$, in the Markov chain for a marked queue, then $Z_k = l \bmod r$ for the other queues associated with the same user, where $r = M/C$.

Proof: We assume that the ratio $r = M/C$ is an integer, and channel j is assigned to user $i = (k - i \cdot r) \bmod M$ in the kth time slot. Therefore, the slot $(i + j \cdot r) \bmod M$ of a frame is assigned to the user i-to-channel j queue. And, for the user i-to-channel j queue, $Z_k = (k - i - j \cdot r) \bmod M$, whereas for the user i-to-channel j' queue, $Z'_k = (k - i - j' \cdot r) \bmod M$.



(a)



(b)

Figure 2: The comparison of analytical (solid curves) and simulation results (circles), $M = 32$, $B = 20$, (a) $C = 8$, (b) $C = 32$.

We then have that

$$\begin{aligned}(Z_k - Z_k') \bmod r &= ((j'-j)\cdot r \bmod M) \bmod r \\ &= ((j'-j)\cdot r \bmod r) \bmod M \\ &= 0,\end{aligned}$$

since $M = 0 \bmod r$. □

Corollary: If $Z_k = 0 \bmod l$ then the marked user is scheduled to transmit over some channel.

From the above theorem we find the average number of non-empty queues associated with the marked user (excluding the marked queue) to be

$$\sum_{\substack{1\le n\le M-1 \\ n=l \bmod r \\ n\ne l}} \sum_{i=1}^{B} \pi(i,0|n).$$

The average number of free channels for which the marked user has packets to send (given that the queue is non-empty) is

$$1 + (1-\sigma) \sum_{\substack{1\le n\le M-1 \\ n=l \bmod r \\ n\ne l}} \sum_{i=1}^{B} \pi(i,0|n).$$

The p robability that the marked user sends a scheduled packet is

$$1 - \sigma \cdot 1_{\{l=0 \bmod r\}}.$$

Finally, the probability that the marked user will decide to transmit a packet over the given channel is given by

$$c(l) = \frac{1-\sigma\cdot 1_{\{l=0 \bmod r\}}}{1+(1-\sigma)\sum_{\substack{1\le n\le M-1 \\ n=l \bmod r \\ n\ne l}} \sum_{i=1}^{B}\pi(i,0|n)}, \quad (12)$$

where $1 \le l \le M-1$.

The probability that the marked user successfully transmits over a given channel is equal to the probability that it chooses to transmit a packet over this channel and no other users do so. The probability that a user with $Z_k = m$ does not access the channel in question is given by

$$\left(1 - c(m)\cdot \sum_{i=1}^{B} \pi(i,0|m)\right).$$

Note that the Markov chains that model queues associated with the same channel take on all values of Z_k except the value l. Hence, the probability of successful transmission is given by the following expression:

$$\gamma(l) = c(l)\cdot \prod_{\substack{m=1 \\ m\ne l}}^{M-1} \left(1 - c(m)\cdot \sum_{i=1}^{B}\pi(i,0|m)\right), \quad (13)$$

where $1 \le l \le M-1$.

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