# On Decoding of Correlative Level Coding Systems with Ambiguity Zone Detection

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*Abstract*—**Decoding of a correlative level coding or partial-response signaling system is discussed in an algebraic framework. A correction scheme in which the quantizer output includes ambiguity levels is proposed. The implementation and algorithm of error correction is discussed in some detail. An optimum design of the quantizer based on Chow's earlier work is discussed. Both analytical and simulation results on the performance of the proposed decoding scheme are presented. An asymptotic expression for the decoding error rate is derived in closed form as a function of the channel signal-to-noise ratio. This is also compared with the conventional bit-by-bit detection method and the maximum-likelihood decoding method recently studied.**

## I. Introduction

A TECHNIQUE in digital data communication developed in recent years is the so-called correlative level coding method (Lender [2]) or the partial-response signaling method (Kretzmer [3]), in which a controlled amount of intersymbol interference is introduced to attain some desired spectral shaping, achieving a high transmission rate at the same time. An equivalent but somewhat different interpretation of the effectiveness of correlative level coding is given in the time domain [4]. The correlative level coding system possesses the property of being relatively insensitive to channel imperfections and also to variations in transmission rate [4], [5]. Recently, it has been pointed out by the present authors [6] that a digital magnetic recording channel can be regarded also as a partial-response channel due to its inherent differentiation in the readback process.

Methods for controlling errors in such a coding or signaling system are discussed in this paper from the standpoint of an algebraic treatment. A mathematical model of a correlative level coding system is reviewed in Section II. Section III describes error-detection schemes which make full use of the inherent redundancy of a correlative level coded output. In these schemes a modulo $m$ detector in the conventional receiver (where $m$ is the number of information sequence levels) is replaced by an inverse filter and a decoder.

In Section IV the algebraic approach is extended to a more general decision scheme, named the ambiguity zone decoding (AZD) method, in which the quantizer makes a soft decision including ambiguity (or rejection) levels. Most of the digits in the ambiguity levels are replaceable by correct values by using the inherent redundancy of the sequence. The correction method developed here is an extension of the null-zone detection method studied by Smith [7].

In Section V an optimum choice of the ambiguity level zones for the quantizer is discussed. The performance of this algebraic decoder is analyzed and is confirmed by computer simulations. Finally, in Section VI, an asymptotic expression for the decoding error rate versus signal-to-noise ratio (SNR) is obtained and compared with the conventional bit-by-bit (BIT) detection method and with the maximum-likelihood decoding (MLD) method recently studied by Kobayashi [8], [9] and Forney [10], [11].

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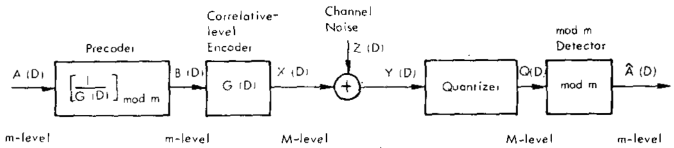


Fig. 1. Conventional system configuration of correlative level coding system.

## II. Mathematical Model [6], [8], [9]

Throughout this paper we deal with a linear discrete system as depicted in Fig. 1. Here the channel has been treated as a digital link comprising the signal generator, modulator, transmission medium, demodulator, filter (possibly an equalizer), and a sampler. A correlative level coding can be realized by shaping any part of the augmented channel just described. However, for the sake of clarity we single out this linear transformation at the transmission side and represent it by a linear discrete filter operating on the information sequence.

Let us represent a sequence by a power series in the delay operator $D$ which is equivalent to the inverse of the $Z$-transform variable $Z$. An information sequence $\{a_k\}$ is thus represented by

$$A(D) = \sum_{k=0}^{\infty} a_k D^k. \tag{1}$$

A correlative level encoder is characterized by a transfer function

$$G(D) = \sum_{i=0}^{N} g_i D^i \tag{2}$$

where the $g_i$ are integers with a greatest common divisor equal to one. We assume without loss of generality that in the pulse-amplitude modulation (PAM) system the input symbol $\{a_k\}$ is chosen from the set of integers $\{0,1,\cdots,m-1\}$.

The information sequence $A(D)$ is first transformed by a precoder with a transfer function $[1/G(D)]_{\text{mod } m}$ into another $m$-level sequence $B(D)$:

$$B(D) \equiv \frac{A(D)}{G(D)}, \qquad \text{modulo } m \tag{3}$$

or, equivalently,

$$G(D)B(D) \equiv A(D), \qquad \text{modulo } m. \tag{4}$$

The precoding is a technique devised by Lender [2] to avoid the error propagation. A generalization to multilevel cases was apparently first done by Gerrish and Howson [12]. In order that the precoder $[1/G(D)]_{\text{mod } m}$ exist, we require that the inverse of $g_0$ exist in the residue class ring modulo $m$. This is assured if $g_0$ and $m$ are relatively prime.

The correlative level encoder transforms $B(D)$ into $X(D)$ according to the relation

$$G(D)B(D) = X(D) \tag{5}$$

or, equivalently,

$$x_k = \sum_{i=0}^{N} g_i b_{k-i}, \qquad \text{for all } k. \tag{6}$$

The encoder output $X(D)$ is a correlated sequence which takes on $M$ different levels, where

$$M = (m-1) \sum_{i=0}^{N} |g_i| + 1.$$

A simple relationship exists between $X(D)$ and $A(D)$ because of (4) and (5):

$$X(D) \equiv A(D), \qquad \text{modulo } m \tag{7}$$

or

$$x_k \equiv a_k, \qquad \text{modulo } m, \quad \text{for all } k. \tag{8}$$

The encoder output $X(D)$ is sent over a channel with an additive noise[1] $Z(D)$, the output of which is denoted by $Y(D)$:

$$Y(D) = X(D) + Z(D). \tag{9}$$

In the conventional receiver structure, the channel output $Y(D)$ is first led to an $M$-level quantizer whose output is denoted by $Q(D)$. If no errors are introduced in the channel and quantizer, then $Q(D) = X(D)$, and the information sequence is recovered simply by performing modulo $m$ operation on $Q(D)$, the output of which is denoted by $\hat{A}(D)$ (Fig. 1). Propagation of errors in the

[1] The noise $Z(D)$ is a time discrete but amplitude continuous random sequence; so is the channel output $Y(D)$.

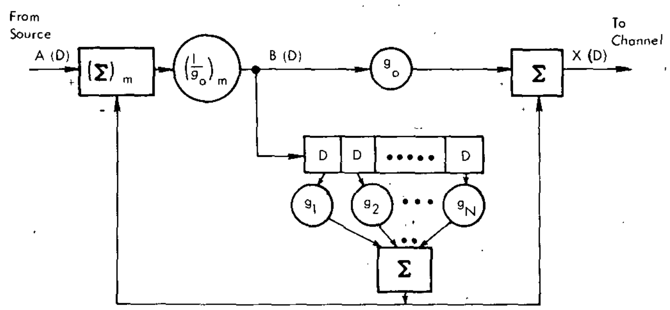


Fig. 2. Combination of precoder and correlative level encoder.

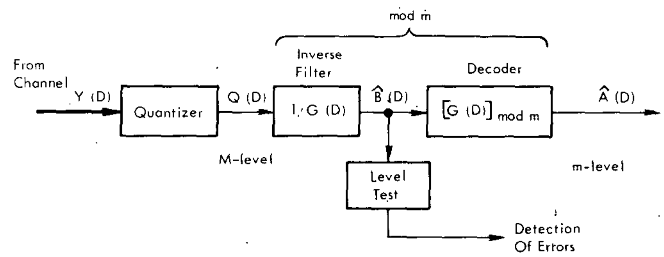


Fig. 3. Receiver structure with error-detection capability.

output sequence $\hat{A}(D)$ due to errors in $Q(D)$ is thus avoided.

Fig. 2 shows an efficient implementation of the precoder and the correlative level encoder in which $(1/g_0)_m$ means the inverse of $g_0$ in the residue class ring modulo $m$, and $(\sum)_m$ denotes the summation in modulo $m$ sense.

## III. Error Detection

### *Detectable Errors*

Although the system shown in Fig. 1 is extremely simple in structure, this detection method is not capable of taking advantage of the inherent redundancy in an $m$-level sequence $X(D)$ and will not detect any errors. To remedy this weakness, Lender [2] proposed an error-detection scheme for the duobinary system. In that scheme the quantizer output $Q(D)$ is monitored by logic circuits to check existence of any unallowable patterns.

In this section we shall describe certain algebraic detection schemes which do not rely on the observation sequence $Q(D)$. Fig. 3 shows a receiver structure which has been reported independently by Gunn and Lombardi [13], the present authors [6], and Forney [10], [11]. The modulo $m$ detector of Fig. 1 is replaced here by the inverse filter with a transfer function $[1/G(D)]$ and the decoder with a transfer function $[G(D)]_{\text{mod } m}$. The decoder performs the inverse operation of the precoder, and its input $\hat{B}(D)$ and output $\hat{A}(D)$ are related by

$$\hat{A}(D) \equiv G(D)\hat{B}(D), \qquad \text{modulo } m. \tag{10}$$

The inverse filter and the decoder together perform an equivalent modulo $m$ operation, namely,

$$\hat{A}(D) \equiv Q(D), \qquad \text{modulo } m \tag{11}$$

and so the objective of avoiding error propagation is still preserved.

If there are no errors in the quantizer output, the inverse filter output $\hat{B}(D)$ is clearly the same as the precoder output $B(D)$. When an error exists in $Q(D)$, it is said to be *detectable* if it could not be generated by a legitimate input $A(D)$ in the absence of errors. Detectable errors can always be detected in the receiving system of Fig. 3 simply by marking the occurrence of any illegitimate coefficients in $\hat{B}(D)$. The validity and the optimality (in the sense of minimal delay) of such a detection scheme is assured by the following theorem.

*Theorem* 1

Any detectable error in the quantizer output $Q(D)$ must result in an inverse filter output $\hat{B}(D)$ which contains a coefficient of level other than the allowable levels $\{0,1,\cdots,m-1\}$. Furthermore, the delay between the occurrence of an error and its detection is the smallest possible.

*Proof*: We first note that the precoder maps an $m$-level sequence $\{a_k\}_0^K$ (a sequence of a finite length $\{a_0,a_1,a_2,\cdots,a_K\}$) onto another $m$-level sequence $\{b_k\}_0^K$ in a one-to-one fashion. Suppose that the quantizer output $\{q_k\}_0^K$ contains some detectable errors. Then there exists no $m$-level sequence $\{a_k\}_0^K$ which would yield this particular sequence $\{q_k\}_0^K$ in the absence of errors. It follows that there is no sequence $\{b_k\}_0^K$ which could produce $\{q_k\}_0^K$ in the absence of errors; i.e.,

$$[G(D)B_K(D)]_K \neq Q_K(D) \tag{12}$$

for any $m$-level sequence $\{b_k\}_0^K$. Here $B_K(D)$ is a $K$th-degree polynomial of $D$ and $[\cdot]_K$ means truncation of a polynomial in the bracket up to degree $K$. Then a sequence $\{\hat{b}_k\}_0^K$ corresponding to

$$\hat{B}_K(D) = \left[\frac{Q_K(D)}{G(D)}\right]_K \tag{13}$$

is not an $m$-level sequence and hence must contain levels other than the allowable levels $\{0,1,\cdots,m-1\}$. To show this, suppose that the inverse filter output of (13) were a legitimate $m$-level sequence. Then

$$[G(D)\hat{B}_K(D)]_K = \left[G(D)\cdot\left[\frac{Q_K(D)}{G(D)}\right]_K\right]_K = Q_K(D) \tag{14}$$

and this result contradicts the assumption (12). The minimum-delay property of error detection follows from the preceding argument, since as soon as an event of (12) holds for some $K$, $\hat{B}_K(D)$ of (13) becomes an illegitimate sequence for that $K$. Q.E.D.

Thus the inverse filter output $\hat{B}(D)$ contains illegitimate digits if any detectable error exists in the sequence $Q(D)$. When $g_0 = 1$, these illegitimate digits are always integers and thus must lie outside the allowable integer levels $\{0,1,\cdots,m-1\}$. On the other hand, when $g_0 \neq 1$, illegitimate levels in $\hat{B}(D)$ correspond, in general, to noninteger numbers. In the next section we shall discuss an error-detection scheme based on Theorem 1 and then introduce a further simplified scheme.

*Error-Detection Schemes*

In this section we shall discuss a more detailed structure of the error-detection system of Fig. 3 and observe its operation. An implementation example is depicted in Fig. 4. Note that the inverse filter and the decoder share the portion $G(D) - g_0$, resulting in a minimum amount of hardware. $\hat{A}(D)$ could be derived directly from $Q(D)_{\bmod m}$ instead of from $\hat{B}(D)$. But the configuration of Fig. 4 leads us naturally to the other scheme, which will be discussed in the sequel.

There are at least two courses of action one may take in response to a detected error. One possible method is to request the retransmission of data tracing back a number of digits from the point where an error is detected. Another possible course of action is to monitor the performance of the system by counting detected errors. Transmission of data is not suspended in this case; consequently, some kind of resetting must be done each time an error is detected. Let us assume that we are dealing with a class of $G(D) = 1 \pm D^N$. In such cases reset pulses (which form a sequence $\{\hat{e}_k\}$) of Fig. 4 are delayed estimates of the error terms and are used to cancel detected errors in the sequence $\{\hat{b}_k\}$ at the earliest possible state in order to prepare the system for the detection of future errors. The error estimate $\{\hat{e}_k\}$ is the minimum number in its magnitude such that $\hat{b}_k - \hat{e}_k$ is an admissible integer. That is, this resetting pulse-generation rule agrees with the minimum-distance decision rule and is given by

$$\hat{e}_k = \begin{cases} \hat{b}_k - (m-1), & \text{for } \hat{b}_k > m-1 \\ \hat{b}_k, & \text{for } \hat{b}_k < 0 \\ 0, & \text{otherwise.} \end{cases} \tag{15}$$

Polynomials $Q(D)$, $\hat{B}(D)$, $\hat{A}(D)$, and $\hat{E}(D)$ are related as follows:

$$\hat{B}(D) = \frac{1}{g_0}[Q(D) - \{G(D) - g_0\}\{\hat{B}(D) - \hat{E}(D)\}] \tag{16}$$

$$\begin{aligned}\hat{A}(D) &\equiv g_0\hat{B}(D) + \{G(D) - g_0\}\{\hat{B}(D) - \hat{E}(D)\}, \quad \text{modulo } m \\ &\equiv Q(D), \quad \text{modulo } m.\end{aligned} \tag{17}$$

Before discussing the operation of the detection circuit, we shall present a different receiver configuration in which the quantizer is embodied in the inverse filter rather than at the head of the receiver (Fig. 5). Note that the threshold range of the quantizer is reduced from $M$ to $(m-1)g_0 + 1$ although the spacing between thresholds is still unity. The principle for the detection procedure and its capability are the same as those of Fig. 4. The output of the subtraction circuit at the head of the re-

TABLE I

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $k$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | ... |
| $a_k$ | 0 | 1 | 1 | 1 | 0 | 1 | 1 | ... |
| $b_k \equiv a_k + b_{k-1}$ (mod 2) | 0 | 1 | 0 | 1 | 1 | 0 | 1 | ... |
| $x_k = b_k - b_{k-1}$ | 0 | 1 | -1 | 1 | 0 | -1 | 1 | ... |
| $z_k$ | 0.1 | 0.2 | -0.1 | -0.4 | (0.6) | -0.3 | 0.2 | ... |
| $y_k = x_k + z_k$ | 0.1 | 1.2 | -1.1 | 0.6 | 0.6 | -1.3 | 1.2 | ... |
| $u_k = y_k + \hat{b}_{k-1} - \hat{e}_{k-1}$ | 0.1 | 1.2 | -0.1 | 0.6 | 1.6 | -0.3 | 1.2 | ... |
| $\hat{b}_k$ | 0 | 1 | 0 | 1 | (2) | 0 | 1 | ... |
| $\hat{e}_k$ | 0 | 0 | 0 | 0 | 1 | 0 | 0 | ... |
| $\hat{a}_k \equiv \hat{b}_k - \hat{b}_{k-1} + \hat{e}_{k-1}$ (mod 2) | 0 | 1 | 1 | 1 | (1) | 1 | 1 | ... |

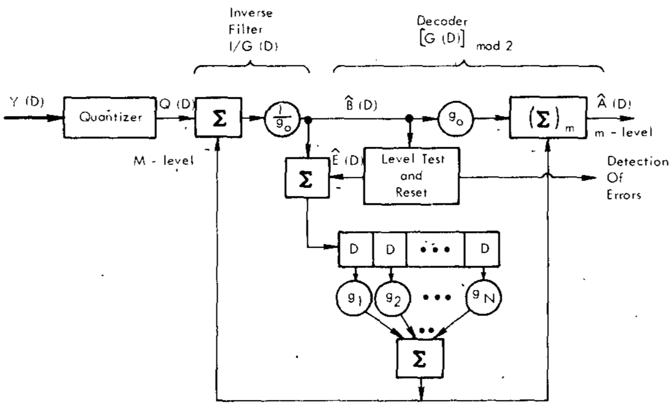


Fig. 4. Error-detection circuit I.

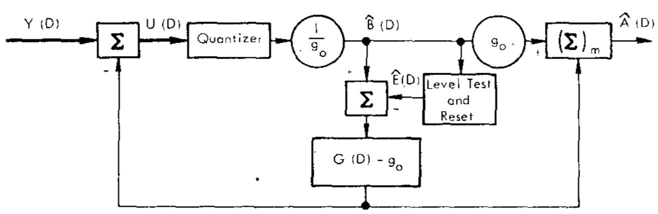


Fig. 5. Error-detection circuit II (quantizer imbedded in inverse filter).

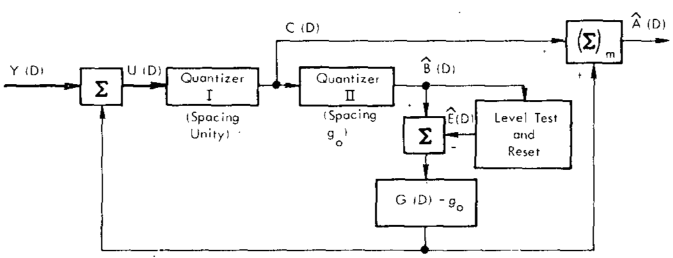


Fig. 6. Error-detection circuit III.

ceiver is now an analog voltage, which we denote by $\{u_k\}$ or $U(D)$:

$$U(D) = Y(D) - \{G(D) - g_0\}\{\hat{B}(D) - \hat{E}(D)\}. \quad (18)$$

The operation of the system will be clarified by providing some examples. Let us first consider a simplest case in which $g_0 = 1$.

*Example* 1: $G(D) = 1 - D$ *and* $m = 2$

Let $G(D) - g_0 = -D$ in Fig. 5. The entries of Table I will be self-explanatory. A strong noise is observed at time $k = 4$. This naturally causes an error in $\hat{b}_4$. However, the error can be detected instantly since $\hat{b}_4 = 2$ is clearly illegitimate. Although $\hat{a}_4 = 1$ is in error, the error does not propagate in the following digits. Note that $\hat{b}_4$ cannot be corrected even though the error is instantaneously detected, since the same sequence $\{\hat{b}_k\}$ would be observed if $b_3 = 0$ (hence $a_3 = 0$, $a_4 = 1$) and $z_3$ were a strong positive noise and $z_4$ were weak, say $z_3 = 0.6$ and $z_4 = 0.4$.

We note that if detection of an error is always followed by a retransmission, then there is no need for resetting and the detection method described earlier is certainly applicable for all $G(D)$ with $(g_0,m) = 1$. However, if uninterrupted transmission is desired, resetting is then in order. The requirement of a simple resetting scheme tends to constrain $G(D)$. We have seen that, for $G(D) = 1 \pm D^N$, the resetting can be simply achieved. For $G(D)$ other than $1 \pm D^N$, we may need more than one observed illegitimate level in $\hat{b}$ sequence to decide how the resetting can be achieved. This implies the need for extra logic circuitry which strongly depends on the specific $G(D)$ chosen.

Next we consider cases with $G(D) = g_0 \pm g_N D^N$, where $g_0 > 1$. In the receiver configuration of Figs. 4 or 5, once an error in quantization is introduced, a noninteger number cycles in the feedback loop unless the error is an integral multiple of $g_0$. Now we modify the structure of Fig. 5 to a more practical one which avoids the occurrence of noninteger numbers. This structure is diagrammatically shown in Fig. 6, in which Quantizer I has the quantization spacings equal to unity. We denote the output of this quantizer by $C(D)$ (or $\{c_k\}$), which is equal to $g_0 B(D)$ unless there exist errors. $C(D)$ is then led to Quantizer II, whose spacing is $g_0 (>1)$. By applying the following quantization rule, the output $\{\hat{b}_k\}$ is assured to be an integer:

$$\hat{b}_k = i, \quad \text{if } ig_0 - \frac{g_0}{2} < c_k \leq ig_0 + \frac{g_0}{2}. \quad (19)$$

The decoding stage is to be modified accordingly, as in Fig. 6; i.e.,

$$\hat{A}(D) = C(D) + \{G(D) - g_0\}\{\hat{B}(D) - \hat{E}(D)\}. \quad (20)$$

*Example* 2: $G(D) = 2 + 3D$ *and* $m = 3$

See Table II. The precoding formula is

$$2b_k + 3b_{k-1} \equiv a_k, \quad \text{modulo } 3 \quad (21)$$

which to equivalent to

$$b_k \equiv 2a_k, \quad \text{modulo } 3. \quad (22)$$

TABLE II

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $k$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| $a_k$ | 0 | 1 | 1 | 2 | 0 | 2 | 1 | 0 | 2 | 1 |
| $b_k \equiv 2 a_k \pmod 3$ | 0 | 2 | 2 | 1 | 0 | 1 | 2 | 0 | 1 | 2 |
| $x_k = 2 b_k + 3b_{k-1}$ | - | 4 | 10 | 8 | 3 | 2 | 7 | 6 | 2 | 7 |
| $z_k$ | 0.1 | -0.1 | -0.7 | 0.2 | -0.3 | 0. | 0.1 | 0.3 | -0.1 | 0.1 |
| $y_k = x_k + z_k$ | 0.1 | 3.9 | 8.3 | 8.2 | 2.7 | 2. | 4.1 | 6.3 | 1.9 | 7.1 |
| $u_k = y_k - 3(\hat{b}_{k-1}-\hat{e}_{k-1})$ | 0.1 | 3.9 | 2.3 | 5.2 | -3.3 | 2. | 4.1 | 0.3 | 1.9 | 4.1 |
| $c_k$ | 0 | 4 | 2 | 5 | -3 | 2 | 4 | 0 | 2 | 4 |
| $\hat{b}_k$ | 0 | 2 | 1 | 2 | -1 | 1 | 2 | 0 | 1 | 2 |
| $\hat{e}_k$ | 0 | 0 | 0 | 0 | -1 | 0 | 0 | 0 | 0 | 0 |
| $\hat{a}_k \equiv c_k + 3(\hat{b}_{k-1}-\hat{e}_{k-1}) \equiv c_k \pmod 3$ | 0 | 1 | 2 | 2 | 0 | 2 | 1 | 0 | 2 | 1 |

As we see from the table there exists a strong noise at $k = 2$, which causes an error in $\{\hat{b}_k\}$ and $\{\hat{a}_k\}$. This error cannot be detected until $k = 4$, at which $\{b_k\}$ takes an outermost level 0. Then $\hat{e}_k = -1$ is generated and is used as a resetting pulse to prepare the system for possible future errors.

We should also remark that if $g_0 > 1$, there are some cases in which the propagating error pattern in the inverse filter $1/G(D)$ quickly dies out. (Take for example, $G(D) = 3 + D$.) In general, if all the roots of

$$G(D) = 0 \tag{23}$$

lie outside the unit circle in the $D$ domain, the propagating error pattern $1/G(D)$ is a converging sequence. When the speed of convergence is fast we may no longer need the precoding operation. An advantage of eliminating precoding is that we can eliminate Quantizer I of Fig. 6, and any noise less than $g_0/2$ in its magnitude does not cause an error. Note also that the decoding stage is now eliminated, i.e., $\hat{A}(D) = \hat{B}(D)$, since no precoding is performed.

## IV. Ambiguity Zone Decoding Method

### *Ambiguity Levels and Error Correction*

The preceding discussion has been based on a system model in which a quantizer makes a hard decision. Such decision schemes, however, discard a substantial amount of information about the reliability originally contained in the received sequence $Y(D)$ or $U(D)$. It is clear that the received digits lying near the boundaries of quantization levels are generally less reliable than those close to ideal signal levels. One way to retain such reliability information of the unquantized signal is to allow the quantizer an option of rejecting a decision for some digits when they lie close to the decision-region boundaries. Most of the rejected digits will be reconstructed later from neighboring digits which will have been received with a higher reliability. An optimum decision scheme with a rejection option has been studied by Chow [14], [15] and has been applied to pattern recognition problems. In coding theory, a rejection is usually referred to as an erasure [16], [17]. The null-zone detection method which Smith

Fig. 7. Decision regions of quantizer with ambiguity zones ($g_0 = 1$).

[7] developed for the duobinary system can be regarded as such a decision scheme.

We shall next extend the algebraic method of error detection discussed in the previous section to include ambiguity levels. Throughout the present section we discuss the system $G(D) = 1 - D$. This system is not only the simplest to analyze, but also represents a class of systems $G(D) = 1 \pm D^N$. We should remark in passing that $G(D) = 1 - D$ with a binary input represents a digital magnetic recording system [6], [8]. All results obtained in the present section are extendable to a class $G(D) = 1 \pm D^N$ with straightforward modifications.

In the absence of noise, the input to the quantizer of Fig. 5 $u_k$ takes on integer levels $\{0,1,\cdots,m - 1\}$. Now the quantizer is modified in such a way that it rejects an instantaneous decision or makes a tentative decision on those digits which fall in ambiguity zones between levels $i$ and $i + 1$, where $i = -1,0,\cdots,m - 1$. Fig. 7 shows a partitioning of the sample space of the quantizer input $U = \{u; -\infty < u < \infty\}$, using this generalized decision rule. The shaded areas with noninteger subscripts $U_{-1/2},U_{1/2},\cdots,U_{m-1/2}$ constitute ambiguity (rejection) level regions. (An optimum choice of decision regions $\{U_n\}$ will be deferred until the next section.) There are several ways to handle these ambiguity digits. In the scheme proposed here, we temporarily assign to the quantizer output $\hat{b}_k$ either one of the two closest integers. For example, an appropriate quantization rule (in general, we can choose randomly one of two closest integers and assign it tentatively to $\hat{b}_k$) will be

$$\hat{b}_k = i, \qquad \text{if } u_k \in U_i \text{ or } U_{i+1/2} \tag{24}$$

where $i = -1,0,1,\cdots,m - 1,m$.

We shall now describe an error-correction procedure based on this generalized quantization rule: the location $k$ of $u_k$ in an ambiguity level is stored temporarily. If the tentative assignment of an integer to the ambiguity digit turns out to be incorrect, it can be found whenever the precoded output reaches the bottom level 0; i.e., $b_{k'} = 0$ for some time $k' \geq k$, since under the rule of (24), the error at $k$, if any, is $-1$ and the propagating error pattern is given by

$$\frac{-D^k}{1-D} = -D^k - D^{k+1}\cdots -D^{k'} - \cdots. \tag{25}$$

In the system $G(D) = 1 + D$, on the other hand, the error-propagation pattern has alternating signs. Thus it is clear that if either $\hat{b}_{k'} > m - 1$ or $\hat{b}_{k'} < 0$ is observed, the error is detected. Once the error is detected, the ambiguity digit can be replaced with a correct value.

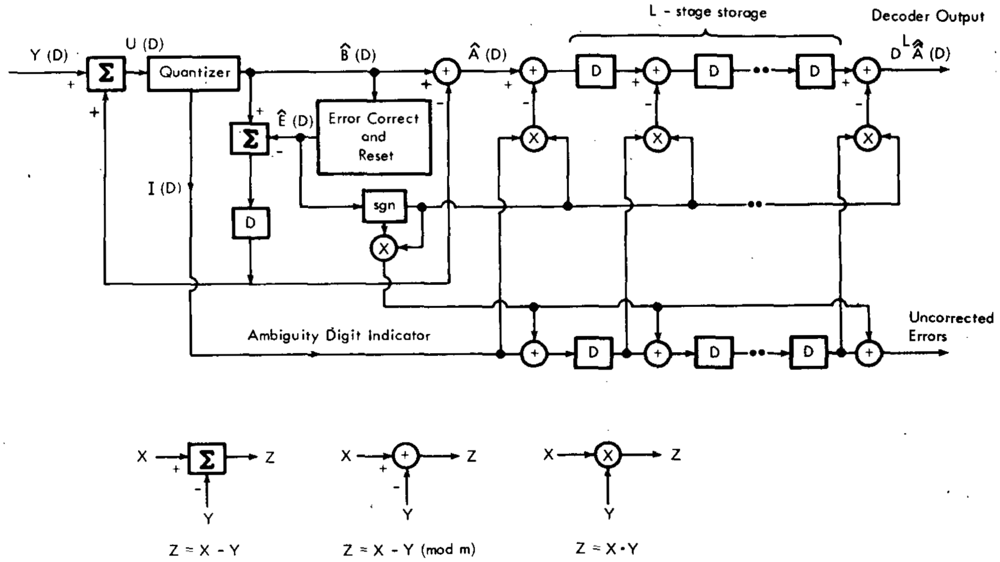


Fig. 8. Ambiguity zone decoder for system $G(D) = 1 - D$.

Fig. 8 shows one possible implementation of the error-correction circuit for a system $G(D) = 1 - D$. (The modification of this receiver structure into a class $G(D) = 1 \pm D^N$ is rather straightforward.) The input to the quantizer is given by

$$u_k = y_k + \hat{b}_{k-1} - \hat{e}_{k-1}. \tag{26}$$

Ambiguity digit indicator sequence $\{i_k\}$ or $I(D)$ is a binary sequence defined by

$$i_k = \begin{cases} 1, & \text{if } u_k \in U_{i+1/2}, \quad i = -1,0,1,\cdots,m-1 \\ 0, & \text{if otherwise.} \end{cases} \tag{27}$$

That is, $i_k$ is 1 if $u_k$ is an ambiguous digit, and is 0 otherwise.

The decoded output $\{\hat{a}_k\}$ is obtained by

$$\hat{a}_k \equiv \hat{b}_k - \hat{b}_{k-1} + \hat{e}_{k-1}, \qquad \text{modulo } m. \tag{28}$$

Here $\hat{e}_k$ in (26) and (28) is a delayed estimate of the propagating error term, as was already defined in Section III. It is generated according to the following rule:

$$\hat{e}_k = \begin{cases} \hat{b}_k - (m-1), & \text{for } \hat{b}_k > (m-1) \\ \hat{b}_k, & \text{for } \hat{b}_k < 0 \\ 0, & \text{otherwise.} \end{cases} \tag{29}$$

If $u_k$ is an ambiguous digit (i.e., $i_k = 1$) and its tentative decision $\hat{b}_k$ is incorrect, then with a high probability we shall observe a nonzero value[2] $\hat{e}_{k'}$ at some time $k'(\geq k)$. Since the error-indicator signal $i_k$ is stored in the memory, signal $\hat{e}_{k'}$ can be subtracted from the erroneously decoded output $\hat{a}_k$; i.e., $i_k$ operates as a gating signal to the error-correction signal $\hat{e}_k$. For the sake of clarity, Fig. 8 shows a circuit which can handle appropriately only a single error in the buffer.

[2] $|\hat{e}_k|$ can be greater than 1 in the case of multiple errors. Discussions on the treatment of multiple errors are given at the end of this section.

The following example will illustrate our scheme most effectively.

*Example* 3: $G(D) = 1 - D$ *and* $m = 2$

Assume that the ambiguity level is defined by

$$U_{i+1/2} = \{u;\, i + 0.4 < u \leq i + 0.6\}. \tag{30}$$

In Table III an ambiguity level is received at time $k = 3$: $u_3 = 0.49$. Thus an erroneous decision $\hat{b}_3 = 0$ is given temporarily and, at the same time, $i_3 = 1$ is stored in the memory. The error cannot be detected until $k = 5$, at which time $\hat{b}_5 = -1$ is observed. The error estimate $\hat{e}_5 = -1$ is then generated and cancels the propagating error pattern in sequence $\hat{B}(D)$. It is also used to replace the ambiguity digit $\hat{b}_3$ with its correct value. The sequence $\{\hat{\hat{a}}_k\}$ is the final decoder output after the correction.

We have seen that an error of plus sign (or minus sign) can be detected as soon as the precoded sequence $b_k$ takes the top-level $m - 1$ (or the bottom-level 0) unless some additional error with the opposite polarity takes place before the arrival of this outermost level. Thus, although the probability of correcting an isolated ambiguity digit approaches one as the buffer length $L$ (see Fig. 8) is increased without bound, the probability of observing two or more ambiguity digits in the buffer storage increases accordingly. In order to handle these multiple ambiguity digits appropriately, the correction circuit of Fig. 8 should be modified as follows. Suppose that $\hat{e}_k \neq 0$ and that there are already more than one ambiguity digits stored in the buffer memory; i.e.,

$$i_{k_1} = i_{k_2} = \cdots = 1 \tag{31}$$

where

$$k \geq k_1 > k_2 \cdots \geq k - L. \tag{32}$$

TABLE III

| $k$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | ... |
|---|---|---|---|---|---|---|---|---|
| $a_k$ | 0 | 1 | 1 | 1 | 0 | 1 | 1 | |
| $b_k \equiv a_k + b_{k-1}$ (mod 2) | 0 | 1 | 0 | 1 | 1 | 0 | 1 | |
| $x_k = b_k - b_{k-1}$ | 0 | 1 | -1 | 1 | 0 | -1 | 1 | |
| $z_k$ | 0.1 | 0.2 | -0.1 | -0.51 | 0.1 | -0.2 | 0.1 | |
| $y_k = x_k + z_k$ | 0.1 | 1.2 | -1.1 | 0.49 | 0.1 | -1.2 | 1.1 | |
| $u_k = y_k + \hat{b}_{k-1} - \hat{e}_{k-1}$ | 0.1 | 1.2 | -0.1 | 0.49 | 0.1 | -1.2 | 1.1 | |
| $\hat{b}_k$ | 0 | 1 | 0 | 0 | 0 | (-1) | 1 | |
| $i_k$ | 0 | 0 | 0 | (1) | 0 | 0 | 0 | |
| $\hat{e}_k$ | 0 | 0 | 0 | 0 | 0 | (-1) | 0 | |
| $\hat{a}_k \equiv \hat{b}_k - \hat{b}_{k-1} + \hat{e}_{k-1}$ | 0 | 1 | 1 | 0 | 0 | 1 | 1 | |
| $\hat{\hat{a}}_k = \hat{a}_k - i_k \operatorname{sgn}\{\hat{e}_{k'}\}$ (mod 2) | 0 | 1 | 1 | (1) | 0 | 1 | 1 | |

Then a correction should be operated on $\hat{a}_{k_1}$ first:

$$a_{k_1} = \hat{a}_{k_1} - \operatorname{sgn}\{\hat{e}_k\}, \qquad \text{modulo } m \tag{33}$$

and $i_{k_1}$ should be reset to zero. The error-estimate signal is increased or decreased by one depending on its polarity:

$$\hat{e}_{k'} = \hat{e}_k - \operatorname{sgn}\{\hat{e}_k\}. \tag{34}$$

If $\hat{e}_{k'}$ is not zero, it means that there are other ambiguity digits yet to be corrected. The next correction should be made on $\hat{a}_{k_2}$; i.e.,

$$a_{k_2} = \hat{a}_{k_2} - \operatorname{sgn}\{\hat{e}_k'\}, \qquad \text{modulo } m \tag{35}$$

and $i_{k_2}$ is set to zero, and so forth.

The correction should be operated in the order of (32), since the signal that indicates an ambiguity digit remains unreset if the error is positive, and hence is rounded down correctly by the rule (24).

## V. Optimum Choice of Ambiguity Zones and Performance Analysis

In this section we shall discuss how the observation space $U = \{u; -\infty < u < \infty\}$ of Fig. 7 should be partitioned into a set of decision regions $\{U_n\}$, based on the results in the statistical decision theory [14], [15], [17], [19]. Analytical and simulation results concerning the performance of the proposed correction scheme will be presented next. Finally, we shall derive an asymptotic expression for the decoding error rate versus SNR in closed form. In the present section, we again limit ourselves to the system $G(D) = 1 - D$, but all results can be extended to the class of $G(D) = 1 \pm D^N$.

If the information sequence $A(D)$ takes on integer levels $\{0,1,\cdots,m-1\}$ independently and equally likely, so does the precoded sequence $B(D)$. Let the probability density function of the additive noise $\{z_k\}$ be denoted by $p_z(z)$. Then the probability of observing the quantizer input $u_k$ when $b_k = i$ is given (assuming the previous $N$ digits of quantizer outputs $\hat{b}_{k-1},\cdots,\hat{b}_{k-N}$ are correct) by $p_z(u_k - i)$, $i = 0,1,\cdots,m-1$. The probability that a digit is decided incorrectly is, therefore

$$E = \frac{1}{m}\sum_{i=0}^{m-1}\sum_{j=0,j\neq i}^{m-1}\int_{U_j} p_z(u-i)\,du$$

$$= \sum_{j=0}^{m-1}\int_{U_j} p_u(u)\,du - \frac{1}{m}\sum_{i=0}^{m-1}\int_{U_i} p_z(u-i)\,du \tag{36}$$

where $p_u(u)$ is the composite probability of the random variable $u$ and is given by

$$p_u(u) = \frac{1}{m}\sum_{i=0}^{m-1} p_z(u-i). \tag{37}$$

Similarly, the rejection rate $R$, i.e., the probability of ambiguous reception, is given by

$$R = \int_{U_R} p_u(u)\,du = 1 - \sum_{i=0}^{m-1}\int_{U_i} p_u(u)\,du \tag{38}$$

where $U_R$ is the union of rejection regions

$$U_R = \bigcup_{i=-1}^{m-1} U_{i+1/2}. \tag{39}$$

As we have already discussed, some of the rejected digits are not replaceable either because of cancellation of propagating error patterns, or because of the finiteness of a delay allowed in the decoder. Let the portion of ambiguity digits which fail to be replaced correctly be denoted by $f(\leq 1)$. Then the total decoding error rate is given by

$$\mathrm{Pe}_{\mathrm{AZD}} = E + f\cdot R \tag{40}$$

where $f$ is not a constant value but a monotonically increasing function of $R$, as will be shown later. Recalling that the optimum decision rule is the one that minimizes the rejection rate $R$ for a given $E$, we are now in a position to apply a theorem due to Chow [14], [15] (see, also, the Appendix) to our problem.

The optimum decision rule stated in Theorem 2 in the Appendix says

$$\text{Reject } u, \quad \text{if} \quad \frac{(1/m)\max_i\{p_z(u-i)\}}{p_u(u)} < 1 - t \tag{41}$$

$$\text{Decide } \hat{b} = j, \text{ if} \quad \frac{(1/m)\max_i\{p_z(u-i)\}}{p_u(u)} = \frac{p_z(u-j)}{mp_u(u)} \geq 1 - t,$$

$$i,j = 0,1,\cdots,m-1 \tag{42}$$

where the parameter $t$ satisfies $1 - 1/m \geq t \geq 0$, and the optimum choice of this parameter is equivalent to the optimum choice of a set of decision regions $\{U_n\}$.

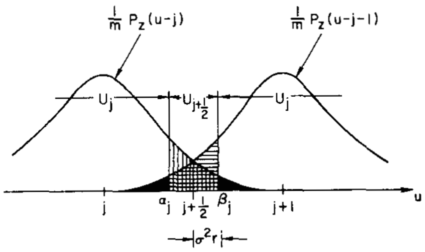


Fig. 9. Decision regions of quantizer for input range $j \leq u \leq j+1$; $\alpha_j = j + 1/2 - \sigma^2 \cdot r$, $\beta_j = j + 1/2 + \sigma^2 \cdot r$

Let the additive noise of the channel be Gaussian with zero mean and variance $\sigma^2$ (normalized by the signal spacing). Then, assuming a reasonably high SNR,[3] we obtain after some manipulation the following expression for optimum rejection regions (Fig. 9):

$$U_{j+1/2} = \{u: j + \tfrac{1}{2} - \sigma^2 \cdot r < u < j + \tfrac{1}{2} + \sigma^2 \cdot r\} \quad (43)$$

where $j = 0,1,\cdots,m-2$ and

$$r = \ln \frac{1-t}{t} \geq 0. \quad (44)$$

The result (43) shows that, for a high SNR, an optimum ambiguity region is symmetrically placed between the integers. Then $E$ and $R$ of (36) and (38) are evaluated:

$$E = \frac{2(m-1)}{m} \cdot Q^- \quad (45)$$

$$R = 2(Q^+ - Q^-) \quad (46)$$

where

$$Q^+ = Q\left(\frac{1}{2\sigma} - \sigma \cdot r\right) \quad (47)$$

$$Q^- = Q\left(\frac{1}{2\sigma} + \sigma \cdot r\right). \quad (48)$$

Here the function $Q(x)$ is defined by

$$Q(x) = \int_x^\infty \frac{1}{(2\pi)^{1/2}} \exp\left(-\frac{y^2}{2}\right) dy. \quad (49)$$

Now we shall obtain an expression for $f$ of (40). As was defined there, $f$ represents the probability that an ambiguity digit is not replaced correctly. For a reasonably high SNR, $f$ consists mainly of the following two terms:

$$f = f_1 + f_2 \quad (50)$$

where $f_1$ is the probability that an ambiguity digit has a negative error (and thus is decoded incorrectly) and that the inverse filter output sequence $\hat{b}_k$ does not exceed the outermost level in $k_0 \leq k \leq k_0 + L$, where $k_0$ is the location of the ambiguity digit and $L$ is the decoder buffer memory size. It is not difficult to see that $f_1$ can be written

$$f_1 = \frac{1}{2}\frac{(m-1)}{m}\left\{\frac{(m-1)}{m}\left(1 - \frac{R}{2}\right)\right\}^L. \quad (51)$$

The term $f_2$ represents twice the probability of an event in which an ambiguity digit with a negative error remains uncorrected and is followed by one with a positive error. When the error due to the wrong replacement of the first ambiguity digit is detected after the second ambiguity digit is received, the second ambiguity digit is also erroneously replaced. This event leads to two errors. It can be shown that $f_2$ is approximated by

$$f_2 = \frac{(m-1)}{m}\left\{\sum_{l=1}^{L}\left\{\frac{(m-1)}{m}(1-R)\right\}^{l-1}\right\}\frac{R}{2}$$

$$= \frac{(m-1)}{m}\frac{R}{2} \cdot \frac{1-(1-1/m)^L(1-R)^L}{1-(1-1/m)(1-R)}. \quad (52)$$

When the decoder buffer size $L$ is sufficiently large, then

$$\lim_{L\to\infty} f_1 = 0 \quad (53)$$

$$\lim_{L\to\infty} f_2 = \frac{(1-1/m)\cdot R/2}{1-(1-1/m)(1-R)}. \quad (54)$$

As can be expected, $f_2$ is a monotonically increasing function of the rejection rate $R$. Now the computation of the decoding error rate is rather straightforward. For a given SNR we calculate the total decoding error rate Pe of (40) for various values of the parameter $r$ (or, equivalently, for $t$) and find the minimum value.

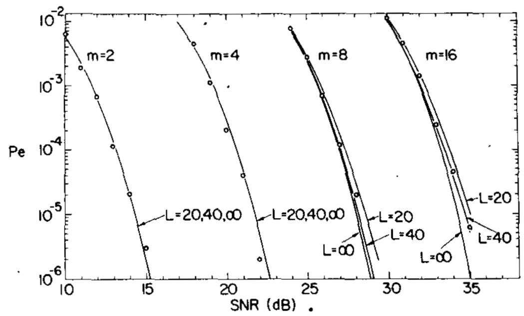


Fig. 10. Decoding error rate versus channel SNR for systems $G(D) = 1 \pm D$ with $m = 2,4,8$, and 16 and $L$ = decoder buffer size. ○—experimental results.

Fig. 10 shows $\mathrm{Pe}_{\mathrm{AZD}}$ versus SNR when an optimum rejection region (i.e., the optimum value of $r$) is used in our decoder for a system $G(D) = 1 \pm D$ with numbers of input signal levels $m = 2,4,8$, and 16. Here the buffer memory size $L = 20$, 40, and infinite are assumed. For $m = 2$ and 4, the difference among these curves for different values of $L$ is unnoticeable. Fig. 10 also shows computer simulation results in which the buffer size $L = 20$ was used. The sample size in the computer simulation was $10^5$ for the range Pe $> 10^{-4}$, and $10^6$ for the range Pe $< 10^{-4}$. The SNR used in Fig. 10 is the average SNR of the channel defined by

$$\eta = \frac{m^2-1}{6\sigma^2}. \quad (55)$$

[3] That is to say,

$$p_u(u) \cong \frac{1}{m}p_z(u-j) + \frac{1}{m}p_z(u-j-1), \qquad \text{for } j \leq u < j+1.$$

Correspondingly, the range of the parameter is changed to $\frac{1}{2} \geq t \geq 0$.

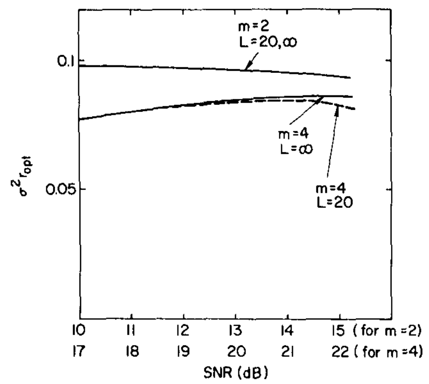


Fig. 11. Ambiguity zone width versus channel SNR for system $G(D) = 1 \pm D$ with $m = 2$ and 4.

We see a satisfactory agreement between the analytical and simulation results.

Fig. 11 shows the optimum ambiguity zone width $\sigma^2 r_{\text{opt}}$ (see (43)) as a function of SNR for $m = 2$ and 4. Note that these curves (for $L = \infty$) approach $\sigma^2 r = 1.5 - \sqrt{2}$ as SNR goes to infinity, i.e., as $\sigma \to 0$. This asymptotic value of the ambiguity zone is derived in Section VI.

## VI. Asymptotic Expression for Decoding Error Rate and Comparison with Other Decoding Methods

We shall now derive an asymptotic (i.e., for a high SNR) expression for the relationship between the decoding error rate and the channel SNR. By approximating the denominator of (54) by unity and using the fact that $Q^+ \gg Q^-$ under a high SNR condition, the decoding error rate of the AZD method can be approximated as follows:

$$\mathrm{Pe}_{\mathrm{AZD}} \cong 2(m-1)\left[\frac{Q^-}{m} + Q^{+^2}\right]. \tag{56}$$

On differentiating $\mathrm{Pe}_{\mathrm{AZD}}$ with respect to the parameter $r$ and setting the result to zero, we have

$$\frac{1}{m}\phi\left(\frac{1}{2\sigma} + \sigma r\right) = 2Q^+ \cdot \phi\left(\frac{1}{2\sigma} - \sigma \cdot r\right) = 0 \tag{57}$$

where $\phi(\cdot)$ is the unit normal distribution function, i.e.,

$$\phi(t) = \frac{1}{(2\pi)^{1/2}}\exp\left(-\frac{t^2}{2}\right). \tag{58}$$

Using the assumption $\sigma \ll 1$ and the approximation formula [20] $Q(x) \cong (1/x)\phi(x)$, $x > 3$, we obtain

$$Q^+ \cong 2\sigma \cdot \phi\left(\frac{1}{2\sigma} - \sigma r\right) \tag{59}$$

$$Q^- \cong 2\sigma \cdot \phi\left(\frac{1}{2\sigma} + \sigma r\right). \tag{60}$$

When (57) holds, we have

$$\frac{Q^-}{m} = 2Q^{+2} \tag{61}$$

$$\mathrm{Pe}_{\mathrm{AZD}} = \frac{3(m-1)}{m} Q^-. \tag{62}$$

Equation (61) indicates that optimum ambiguity zones for a high SNR are such that $E = 2fR$, i.e., a true error is as probable as two ambiguity digits erroneously decoded. The optimum value for the parameter $r$ is obtained from (57), yielding

$$r_0 = \frac{3}{2\sigma^2} - \frac{1}{\sigma}\left(\frac{2}{\sigma^2} - \ln\frac{\pi}{8m^2\sigma^2}\right)^{1/2}. \tag{63}$$

Hence the argument of $Q^-$ is given by

$$\frac{1}{2\sigma} + \sigma \cdot r_0 = \frac{2}{\sigma} - \left(\frac{2}{\sigma^2} - \ln\frac{\pi}{8m^2\sigma^2}\right)^{1/2}$$

$$\cong (2 - \sqrt{2})\frac{1}{\sigma}. \tag{64}$$

Using the channel SNR defined by (55), we finally obtain the expression we have been seeking:

$$\mathrm{Pe}_{\mathrm{AZD}} \cong 3\left(1 - \frac{1}{m}\right) Q\left[2(\sqrt{2} - 1)\left(\frac{3\eta}{m^2 - 1}\right)^{1/2}\right]. \tag{65}$$

An asymptotically (i.e., for a small $\sigma$) optimal choice of the ambiguity zone $U_{i+1/2}$ is

$$U_{i+1/2} = \{u;\, i + \tfrac{1}{2} - \sigma^2 r_0 < u \le i + \tfrac{1}{2} + \sigma^2 r_0\} \tag{66}$$

where $\sigma^2 r_0$ is given from (63) by

$$\sigma^2 r_0 = \frac{3}{2} - \left(2 - \sigma^2 \ln\frac{\pi}{8m^2\sigma^2}\right)^{1/2}. \tag{67}$$

In fact, (67) agrees very well with curves drawn for $L = \infty$ in Fig. 11.

Now we compare the asymptotic expression (65) with other decoding methods. In the conventional BIT detection method (see Fig. 1) the error rate is given by [5]

$$\mathrm{Pe}_{\mathrm{BIT}} = 2\left(1 - \frac{1}{m^2}\right) Q\left(\frac{3\eta}{2(m^2 - 1)}\right)^{1/2}. \tag{68}$$

An asymptotic expression for the decoding error rate for the MLD method has been recently derived by one of the present authors [8], [9] and Forney [11] independently:

$$\mathrm{Pe}_{\mathrm{MLD}} \cong 4(m-1) Q\left(\frac{3\eta}{m^2 - 1}\right)^{1/2}. \tag{69}$$

Fig. 12 shows plots of (65), (68), and (69). We see that the performance of the error-correction method lies between the BIT detection method and the MLD method. If we ignore constant factors the decoding error rates for three methods are given in the form

$$\mathrm{Pe} \sim Q\left(\frac{3\eta}{\alpha(m^2 - 1)}\right)^{1/2}$$

where

$$\alpha = \begin{cases} 1, & \text{for MLD} \\ \dfrac{3 + 2\sqrt{2}}{4} = 1.457, & \text{for AZD} \\ 2, & \text{for BIT.} \end{cases} \tag{70}$$

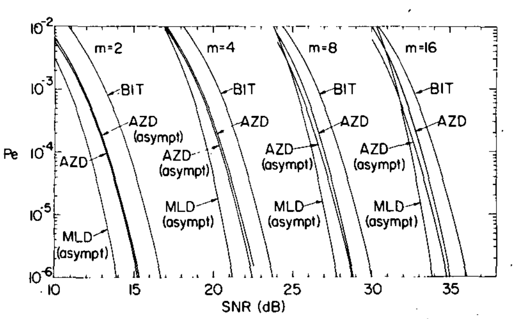


Fig. 12. Comparison of BIT, AZD, and MLD methods.

It is interesting to observe[4] that the loss factor $4/(3 + 2\sqrt{2}) = 4(3 - 2\sqrt{2})$ of the AZD method with respect to the MLD method appeared in earlier work by Forney [18, eq. (44)], where he discussed erasure-and-error decoding of a group code when the channel noise is white Gaussian and modulation is binary antipodal signaling. Although somewhat inferior to the MLD in its performance, the AZD method possesses an advantage over the MLD in its simple implementation. The number of quantization levels is, in general, much smaller than that required in the MLD [8], [9]. Furthermore, in the MLD method the number of "states," which determines the decoder complexity, is[5] $mN$ for a system $G(D) = 1 \pm D^N$. Thus the MLD algorithm tends to require a significant amount of computation effort and memory requirement when the number of signal levels increases. The AZD method will be more attractive in that respect.

## VII. Concluding Remarks

We have discussed in preceding sections the general problem of detection and correction of errors in a correlative level coding system using ambiguity zone detection. The application of this method to a class of systems with $G(D) = 1 \pm D^N$ is analyzed in greater detail, and its performance is shown to be somewhat inferior to the maximum-likelihood decoding method but superior to the bit-by-bit detection method. We have recently learned that Forney [11], in his independent work, also proposed an error-correction scheme by tracking illegitimate levels in the inverse filter output, except that the received-signal level is considered as a continuous quantity. The additional quantization in the form of ambiguity zones suggested in this paper can be viewed as an attempt to obtain simpler system design at the expense of moderate performance degradation.

## Appendix

### Optimum Decision Rule with Reject Option [14], [15], [17], [19]

We shall derive an optimum decision rule of (41) and (42). We use a somewhat different approach from the original proof given by Chow [14]. As one may realize from the following proof, the error and rejection tradeoff in this generalized decision rule is analogous to the tradeoff between two types of error in a simple binary hypothesis testing problem. The derivation of the optimum decision rule is similar to the well-known Neyman–Pearson lemma [19].

*Definition*: We say that a decision rule is *optimum* if, for a given rejection rate $R \leq \alpha$, it minimizes the error rate $E$.

*Theorem 2*

Let $H_i$, $i = 1,2,\cdots,n$, be $n$ different hypotheses, and let $\pi_i$ be the *a priori* probability of hypothesis $H_i$. Let $y$ be the observable (possibly a vector, or a continuous function of time), and let the conditional probability of observing $y$ under hypothesis $H_i$ be denoted by $p(y \mid H_i)$. Then the optimum decision rule is determined by partitioning the observation space (sample space) $Y$ into a set of disjoint subspaces $Y_1,Y_2,\cdots,Y_n$ and $Y_R$ as follows:

$$Y_i = \{y: \max_j \{p(H_j \mid y)\} = p(H_i \mid y) \geq 1 - \lambda\}, \qquad i,j = 1,2,\cdots,n \quad (71)$$

$$Y_R = \{y: \max_j \{p(H_j \mid y)\} < 1 - \lambda\}. \quad (72)$$

Here $Y_R$ is the rejection region and $p(H_j \mid y)$ is the posterior probability of hypothesis $H_j$ conditioned on the observable $y$ and is given by

$$p(H_j \mid y) = \frac{\pi_j p(y \mid H_j)}{\sum_{i=1}^{n} \pi_i p(y \mid H_i)}. \quad (73)$$

The threshold constant $\lambda$ satisfies $1 - 1/n \geq \lambda \geq 0$ and is determined from the constraint $R \leq \alpha$; i.e.,

$$\sum_{i=1}^{n} \int_{Y_R} \pi_i p(y \mid H_i)\, d\mu(y) \leq \alpha \quad (74)$$

where $\mu(\cdot)$ is the measure defined in the space $Y$.

*Proof*: Let $Y_1,Y_2,\cdots,Y_n$ and $Y_R$ be any set of disjoint and exhaustive subspaces of the sample space $Y$. Then the error rate $E$ and the rejection rate $R$ are given by

$$E = \sum_{i=1}^{n} \sum_{j=1, j\neq i}^{n} \int_{Y_j} \pi_i p(y \mid H_i)\, d\mu(y)$$

$$= \sum_{j=1}^{n} \int_{Y_j} p(y)\, d\mu(y) - \sum_{i=1}^{n} \int_{Y_i} \pi_i p(y \mid H_i)\, d\mu(y) \quad (75)$$

$$R = \int_{Y_R} p(y)\, d\mu(y) = 1 - \sum_{i=1}^{n} \int_{Y_i} p(y)\, d\mu(y) \quad (76)$$

where $p(y)$ is the probability density function of $y$

$$p(y) = \sum_{i=1}^{n} \pi_i p(y|H_i). \quad (77)$$

We want to minimize $E$ under the constraint $R \leq \alpha$. Following the usual procedure of Lagrangian coefficient

[4] This similarity was pointed out by one of the reviewers.

[5] The decoder for $G(D) = 1 + D^N$ is composed of $N$ copies (in the interleaved form) of the decoder for $1 \pm D$. Note that the number of states is $m^N$ for a general $G(D)$ of the $N$th degree.

method, we define $J$ by

$$J = E + \lambda \cdot \{R - \alpha\}, \qquad \lambda \geq 0. \tag{78}$$

On substituting (75) and (76) into (78), one obtains

$$J = \lambda(1-\alpha) - \sum_{i=1}^{n} \int_{Y_i} \{\pi_i \cdot p(y \mid H_i) - (1-\lambda)\cdot p(y)\}\, d\mu(y). \tag{79}$$

Let us define the index $m = m(y)$ by

$$\pi_m p(y \mid H_m) = \max_i \{\pi_i p(y \mid H_i)\}. \tag{80}$$

Then, clearly, (79) is bounded below as follows:

$$\begin{aligned} J &\geq \lambda(1-\alpha) - \int_{Y-Y_R} \{\pi_m p(y \mid H_m) - (1-\lambda)p(y)\}\, d\mu(y) \\ &= \lambda(1-\alpha) - \int_{Y-Y_R} p(y)\{p(H_m \mid y) - (1-\lambda)\}\, d\mu(y) \end{aligned} \tag{81}$$

where $p(H_m \mid y)$ is defined by (73).

Since $p(y) \geq 0$, for all $y$, the last expression of (81) is minimized when the rejection region $Y_R$ is determined by the following rule.

$$\text{If } p(H_m \mid y) > 1-\lambda, \quad \text{then } y \in Y - Y_R, \quad m = 1,2,\cdots,n. \tag{82}$$

$$\text{If } p(H_m \mid y) \leq 1-\lambda, \quad \text{then } y \in Y_R. \tag{83}$$

The equality in (81) holds when the acceptance region $Y - Y_R$ is partitioned into a set of $Y_i$ according to

$$Y_i = \{y: p(H_m \mid y) = p(H_i \mid y) > 1-\lambda\}, \quad i = 1,2,\cdots,n. \tag{84}$$

The parameter $\lambda$ is chosen in such a way that (74) is satisfied. Thus, for $y \in Y_R$, we obtain the following expression from (73) and (83):

$$1-\lambda \geq p(H_m \mid y) = \frac{\pi_m p(y \mid H_m)}{p(y)} \geq \frac{1}{n}\frac{\sum_{i=1}^{n} \pi_i p(y \mid H_i)}{p(y)} = \frac{1}{n}. \tag{85}$$

Therefore, $\lambda$ is bounded by the following relation:

$$1 - \frac{1}{n} \geq \lambda \geq 0. \tag{86}$$

Q.E.D.


## Acknowledgment

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