A unified analysis for Cramer-Rao Lower Bound for non-line-of-sight geolocation

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Abstract — **Non-line-of-sight (NLOS) geolocation becomes an important issue with the fast development of mobile communications in recent years. Several methods have been proposed to mitigate NLOS effects in geolocation. However, there has been no systematic study reported on how the existing methods are related to one another and what is the best geolocation accuracy that we may possibly achieve. As first step to answer these questions, here we present a unified treatment to obtain Cramer-Rao Lower Bound (CRLB) for time-of-arrival (TOA), maximum likelihood estimation (MLE) and signal strength (SS) based positioning methods (to be defined below). Its physical interpretation is discussed and is very helpful to establish connection among the three methods. It can be shown that the CRLB of MLE is equivalent to that of first estimating time delay with a matched filter and then perform TOA positioning. One interesting observation is that the CRLB depends only on the signals obtained by line-of-sight (LOS) base stations and NLOS signals are completely discarded. Furthermore, we notice that a similar conclusion can be draw without the explicit expressions of NLOS geolocation. Thus we extend the current problem to a broader formulation of the CRLB of parameter estimation with unwanted parameters.**

I. INTRODUCTION

Geolocation in an non-line-of-sight (NLOS) environment is an important issue in mobile communications, and is receiving a considerable attention in recent years. Several empirical methods [1]–[5] have been proposed to mitigate NLOS effects in geolocation. However, two fundamental questions are still not clear: what is the best achievable geolocation accuracy in NLOS environment and how the different methods are connected.

In this paper, we manage to answer these questions under two assumptions: one is that no prior information on NLOS induced paths or the mobile station (MS) position is available; the other is that only a single (line-of-sight (LOS) or NLOS) path between a given mobile station (MS) and base station (BS) pair are allowed. We extend the analysis to the scenario when the prior knowledge is considered in [6]. We hope that thorough understanding of the single path case will shed light on our investigation into intricate multipath situations. Here we develop a new unified analysis of the Cramer-Rao Lower Bound (CRLB) for NLOS geolocation, which generalizes time-of-arrival (TOA), maximum likelihood estimation (MLE) and signal strength (SS) based methods. Its physical interpretation suggests that the CRLB is determined solely by line-of-sight (LOS) signals, and the NLOS parts ought to be totally ignored. Within this framework, an obvious relation between the optimum MLE approach and the conventional TOA methods arises: the CRLB of the MLE is equivalent, at high signal-to-noise ratio (SNR), to that of a geolocation algorithm, which can be implemented in two consecutive steps: first, to estimate the TOA's at matched filter outputs in relevant BS's; second, to determine the MS location based on the conventional least square error method using the TOA data from at least three BS's. An immediate significance of this result is that the (presumably) complicated MLE solution is now decomposed into the two simple and practical components. In the end, we consider a general format of the CRLB of parameter estimation with unwanted parameters, within which the NLOS geolocation problem acts as a special case.

The rest of the paper is structured as follows. In Section II, we investigate a unified approach for evaluating the CRLB for NLOS geolocation with TOA, MLE and SS based methods. By an explicit formula, its physical meanings are interpreted. We then explore the relation of the MLE and TOA based positioning in Section III. Section IV characterizes a generalization of the CRLB of NLOS geolocation. We state a brief conclusion in Section V.

II. CRLB FOR NLOS GEOLOCATION

The Cramer-Rao inequality gives a lower bound for error variances of any unbiased estimates of some unknown parameters [7]. Denote $\hat{\underline{\theta}}$ as an estimate of the vector of parameters $\underline{\theta}$. Its covariance matrix is $\mathrm{Cov}_{\underline{\theta}}(\hat{\underline{\theta}}) = E_{\underline{\theta}}\left\{(\hat{\underline{\theta}}-\underline{\theta})(\hat{\underline{\theta}}-\underline{\theta})^*\right\}$, where $E_{\underline{\theta}}\{\cdot\}$ stands for an expected value conditioned on $\underline{\theta}$ and symbol "*" for complex conjugate and transpose. Let $f_{\underline{\theta}}(\underline{r})$ be the probability density function (p.d.f.) of observations $\underline{r}$ conditioned on $\underline{\theta}$. Its *Fisher information matrix* is given by

$$\mathbf{J}_{\underline{\theta}} = E_{\underline{\theta}}\left\{\frac{\partial}{\partial\underline{\theta}}\log f_{\underline{\theta}}(\underline{r})\cdot\left(\frac{\partial}{\partial\underline{\theta}}\log f_{\underline{\theta}}(\underline{r})\right)^*\right\}. \quad (1)$$

The CRLB is then expressed as

$$\mathrm{Cov}_{\underline{\theta}}(\hat{\underline{\theta}}) \geq \mathbf{J}_{\underline{\theta}}^{-1}. \quad (2)$$

Let $\mathcal{B} = \{1, 2, \cdots, B\}$ be the set of indices of base stations, which are located at $\left\{\underline{p_b} = (x_b, y_b),\ b \in \mathcal{B}\right\}$. Let $\mathcal{M} =$

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$\{k_1, k_2, \cdots, k_M\}$ be the set of the BS's that receive NLOS signals from a MS of concern. Thus the complement $\mathcal{L} = \mathcal{B} \backslash \mathcal{M}$ is the set of LOS BS's with its cardinality $|\mathcal{L}| = B - M$. We can assume $\mathcal{M} = \{1, 2, \cdots, M\}$ without loss of generality. Denote τ_b as the time delay of the signal between base station b (BS_b) and the MS. Two sets of parameters to be estimated are the MS position $\underline{p} = (x, y)$ and NLOS propagation induced path lengths $\underline{l} = (l_1, l_2, \cdots, l_M)$. We divide various NLOS geolocation methods into the following three classes:

Class 1. TOA based methods [1]

The time delay estimates (i.e., TOA's), which consist of the LOS delay and possible extra path delay $\underline{l}$, are obtained beforehand. Specifically, the time delay estimates are

$$\rho_b = \tau_b + \eta_b, \quad \text{for } b \in \mathcal{B}, \tag{3}$$

with

$$\tau_b = \frac{1}{c}\left(\sqrt{(x_b - x)^2 + (y_b - y)^2} + l_b\right), \tag{4}$$

where the measurement error η_b is approximated as a Gaussian random variable with $\mathcal{N}(0, \sigma_b^2)$, $c = 3 \times 10^8$ m/s is the speed of light and $l_b = 0$ for $b \in \mathcal{L}$. This model becomes accurate when the TOA's are acquired with the matched filter approach at high SNR. The MS position is then derived from the set of $\{\rho_b, \ b \in \mathcal{B}\}$.

Class 2. MLE methods

In contrast to Class 1, this class starts directly from noisy observations at BS receivers, i.e.,

$$r_b(t) = A_b s(t - \tau_b) + n_b(t), \quad \text{for } b \in \mathcal{B}, \tag{5}$$

where $s(t)$ and A_b is a known signal waveform and the signal amplitude respectively, and $n_b(t)$'s are complex-valued white Gaussian noise processes with spectral density $N_0/2$. We try to obtain an MLE of $\underline{p} = (x, y)$ by processing $\{r_b(t), \ b \in \mathcal{B}\}$.

Class 3. SS based methods [1]

In some circumstances, signal strength is the major quantity that contains the information regarding a propagation path, or equivalently τ_b, while τ_b in the received signal waveform is not distinguishable. The received signal is then represented as

$$r_b(t) = \frac{A}{\tau_b^{\epsilon}} s(t) + n_b(t), \quad \text{for } b \in \mathcal{B}, \tag{6}$$

where the constant $\epsilon > 1$ is the propagation loss parameter that may be empirically determined. $n_b(t)$ is same as defined in Eq. (5). We wish to estimate the MS position from $\{r_b(t), \ b \in \mathcal{B}\}$.

We define an $(M+2)$-dimensional vector $\underline{\theta} = (\underline{p}, \ \underline{l})$. It is clear that the parameter of our interest is only $\underline{p} = (x, y)$. The p.d.f's of the observations $\underline{r}$ conditioned on $\underline{\theta}$ are

$$f_{\underline{\theta}}\left[\underline{\rho}\right] \propto \prod_{b=1}^{B} \exp\left\{-\frac{1}{2\sigma_b^2}(r_b - \tau_b)^2\right\}, \quad \text{for TOA}, \tag{7}$$

$$f_{\underline{\theta}}\left[\underline{r}(t)\right] \propto \prod_{b=1}^{B} \exp\left\{-\frac{1}{N_0}\int |r_b(t) - A_b s(t - \tau_b)|^2 \, dt\right\},$$

for MLE (8)

$$f_{\underline{\theta}}\left[\underline{r}(t)\right] \propto \prod_{b=1}^{B} \exp\left\{-\frac{1}{N_0}\int \left|r_b(t) - \frac{A}{\tau_b^{\epsilon}} s(t)\right|^2 dt\right\},$$

for SS. (9)

Thus, by casting the NLOS geolocation as a multi-parameter estimation problem, we can evaluate the CRLB for the parameter vector $\underline{\theta}$. Note that the conditional p.d.f.'s in Eqs. (7)–(9) are functions of τ_b's, which in turn are functions of the parameters in $\underline{\theta}$ as stated in Eq. (4). Thus with *chain rule*, $\mathbf{J}_{\underline{\theta}}$ in Eq. (1) becomes

$$\mathbf{J}_{\underline{\theta}} = \mathbf{H} \cdot \mathbf{J}_{\underline{\tau}} \cdot \mathbf{H}^*, \tag{10}$$

where

$$\mathbf{H} = \begin{pmatrix} \frac{\partial \tau_1}{\partial x} & \cdots & \frac{\partial \tau_M}{\partial x} & \cdots & \frac{\partial \tau_B}{\partial x} \\ \frac{\partial \tau_1}{\partial y} & \cdots & \frac{\partial \tau_M}{\partial y} & \cdots & \frac{\partial \tau_B}{\partial y} \\ \frac{\partial \tau_1}{\partial l_1} & \cdots & \frac{\partial \tau_M}{\partial l_1} & \cdots & \frac{\partial \tau_B}{\partial l_1} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \frac{\partial \tau_1}{\partial l_M} & \cdots & \frac{\partial \tau_M}{\partial l_M} & \cdots & \frac{\partial \tau_B}{\partial l_M} \end{pmatrix}, \tag{11}$$

an $(M+2) \times B$ matrix, and $\mathbf{J}_{\underline{\tau}}$ is the *Fisher information matrix* of B observables:

$$\mathbf{J}_{\underline{\tau}} = E_{\underline{\tau}}\left[\frac{\partial}{\partial \underline{\tau}} \log f_{\underline{\theta}} \cdot \left(\frac{\partial}{\partial \underline{\tau}} \log f_{\underline{\theta}}\right)^*\right]. \tag{12}$$

Substituting Eq. (4) into Eq. (11), we have

$$\mathbf{H} = \begin{pmatrix} \mathbf{H}_1 & \mathbf{H}_2 \\ \frac{1}{c}\mathbf{I}_M & \mathbf{0} \end{pmatrix}, \tag{13}$$

where $\mathbf{I}_M$ is an identity matrix of order M, $\mathbf{H}_1$ and $\mathbf{H}_2$ are $2 \times M$ and $2 \times (B - M)$ matrices, respectively, given by

$$\mathbf{H}_1 = \frac{1}{c} \cdot \begin{pmatrix} \cos\phi_1 & \cdots & \cos\phi_M \\ \sin\phi_1 & \cdots & \sin\phi_M \end{pmatrix},$$

$$\mathbf{H}_2 = \frac{1}{c} \cdot \begin{pmatrix} \cos\phi_{M+1} & \cdots & \cos\phi_B \\ \sin\phi_{M+1} & \cdots & \sin\phi_B \end{pmatrix},$$

and angle ϕ_b is determined by

$$\tan^{-1}\phi_b = \frac{x - x_b}{y - y_b}.$$

Note that the quantity in Eq. (13) with subscript "1" is related to NLOS BS's, while subscript "2" is to LOS counterparts. Similar notation is applied to the following equations.

With Eqs. (7)-(9), Eq. (12) yields

$$\mathbf{J}_{\underline{\tau}} = \begin{pmatrix} \mathbf{\Sigma}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{\Sigma}_2 \end{pmatrix}, \tag{14}$$

where $\mathbf{\Sigma}_1$ and $\mathbf{\Sigma}_2$ are diagonal matrices of order M and $(B - M)$, respectively, given by

$$\mathbf{\Sigma}_1 = \begin{pmatrix} \lambda_1 & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \lambda_M \end{pmatrix}, \quad \text{and}$$

$$\mathbf{\Sigma}_2 = \begin{pmatrix} \lambda_{M+1} & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \lambda_B \end{pmatrix}.$$

Their entries differ according to the types of models:

$$\lambda_b = \begin{cases} \frac{1}{\sigma_b^2}, & \text{for TOA}, \\ 8\pi^2\beta^2 \cdot R_b, & \text{for MLE}, \\ \frac{2\epsilon^2}{\tau_b^2} \cdot R_b, & \text{for SS}, \end{cases} \tag{15}$$

where R_b is the SNR of the received signal at BS_b, i.e.,

$$R_b = \frac{\int |A_b s(t)|^2 dt}{N_0} \quad \text{for MLE, and}$$
$$R_b = \frac{\int |\frac{A}{\tau_b^\epsilon} s(t)|^2 dt}{N_0} \quad \text{for SS.} \tag{16}$$

The effective bandwidth of the signal waveform β is determined by

$$\beta^2 = \int f^2 |S(f)|^2 df,$$

where $S(f)$ is the Fourier transform of $s(t)$. One assumption made here is the normalization condition $\int |s(t)|^2 dt = 1$.

Substitute Eqs. (13) and (14) into (10), it is straightforward to obtain

$$\mathbf{J}_{\underline{\theta}} = \begin{pmatrix} \mathbf{H}_1\mathbf{\Sigma}_1\mathbf{H}_1^* + \mathbf{H}_2\mathbf{\Sigma}_2\mathbf{H}_2^* & \frac{1}{c}\mathbf{H}_1\mathbf{\Sigma}_1 \\ \frac{1}{c}\mathbf{\Sigma}_1\mathbf{H}_1^* & \frac{1}{c^2}\mathbf{\Sigma}_1 \end{pmatrix}. \tag{17}$$

We notice that $\mathbf{J}_{\underline{\theta}}$ depends on BS's in both $\mathcal{M}$ (NLOS) and $\mathcal{L}$ (LOS). However, we show in Appendix that

$$\text{CRLB} = \left[\mathbf{J}_{\underline{\theta}}^{-1}\right]_{nn} = \left[(\mathbf{H}_2\mathbf{\Sigma}_2\mathbf{H}_2^*)^{-1}\right]_{nn}, \quad \text{for } n = 1, 2. \tag{18}$$

Hence the CRLB for the MS position is just relied on submatrices $\mathbf{H}_2$ and $\mathbf{\Sigma}_2$, which involve only signals received by LOS stations. The contribution from the NLOS signals does not count.

Because of the inherent similarities, it is adequate to confine our discussion to the explicit formula of the MLE model. Thus, with Eq. (18), the mean squared Euclidean distance between the true MS position $\underline{p} = (x, y)$ and its estimate $\underline{\widehat{p}} = (\hat{x}, \hat{y})$ is lower bounded by

$$E_{\underline{\theta}}\left[(\underline{p} - \underline{\widehat{p}})^2\right] = E_{\underline{\theta}}\left[(x - \hat{x})^2\right] + E_{\underline{\theta}}\left[(y - \hat{y})^2\right]$$
$$\geq \frac{c^2}{\alpha} \cdot \frac{\sum_{b\in\mathcal{L}} R_b}{\sum\sum_{b_1,b_2\in\mathcal{L}} R_{b_1}R_{b_2}\sin^2(\phi_{b_1} - \phi_{b_2})}, \tag{19}$$

where $\alpha = 8\pi^2\beta^2$.

In estimating the path length l_m of the signal received at NLOS base station m, we find the following lower bound of the estimate error

$$E_{\underline{\theta}}\left[(l_m - \widehat{l}_m)^2\right] \geq \left[\mathbf{J}_{\underline{\theta}}^{-1}\right]_{m+2,m+2}$$
$$= \frac{c^2}{\alpha R_m} \cdot \frac{\sum\sum_{b_1,b_2\in\mathcal{L}\cup\{m\}} R_{b_1}R_{b_2}\sin^2(\phi_{b_1} - \phi_{b_2})}{\sum\sum_{b_1,b_2\in\mathcal{L}} R_{b_1}R_{b_2}\sin^2(\phi_{b_1} - \phi_{b_2})},$$
$$\geq \frac{c^2}{\alpha R_m}, \quad \text{for } m \in \mathcal{M}. \tag{20}$$

Now we are in a position to provide a physical interpretation of Eqs. (19) and (20).

First, as expected, $E_{\underline{\theta}}\left[(\underline{p} - \underline{\widehat{p}})^2\right]$ depends only on the LOS signals. This implies that when trying to estimate an MS position in NLOS environment, the performance should be the same for the algorithm of separating and then rejecting NLOS signals as that of actually estimating the NLOS induced path length l_m's along with $\underline{p} = (x, y)$. The former usually requires less computational complexity. If we need to estimate an l_m for other purpose, such as channel estimation, a similar conclusion from Eq. (20) is drawn: the other $(M - 1)$ NLOS BS's do not help to improve the estimation accuracy of $\widehat{l}_m$. Only the signal received by $(B - M)$ LOS BS's and the NLOS signal at the specific BS_m matter.

Secondly, $E_{\underline{\theta}}\left[(\underline{p} - \underline{\widehat{p}})^2\right]$ is inversely proportional to the square of effective bandwidth, β^2, as previously known in radar estimation (see e.g., [9]).

Thirdly, the accuracy is influenced by the geometric relation among the LOS BS's, only through sine functions of the angle differences $(\phi_{b_1} - \phi_{b_2})$, $b_1, b_2 \in \mathcal{L}$, seen by the MS. It becomes infinitely large when all LOS BS's and the MS happen to lie on a straight line, i.e.,

$$E_{\underline{\theta}}\left[(\underline{p} - \underline{\widehat{p}})^2\right] \longrightarrow +\infty, \quad \text{as} \quad \phi_{b_1} - \phi_{b_2} = 0, \pi, \quad \text{for } b_1, b_2 \in \mathcal{L}.$$

Fortunately, such is rarely the case in reality.

III. Relation between MLE and TOA based approaches

Formulating geolocation as an MLE problem directly applied to the received waveforms has various advantages, e.g., its theoretical analysis naturally relates to some important signal parameters, such as the SNR and the effective bandwidth efficiency (β). However, its direct mathematical solution often demand to evaluate a complex optimization problem, while the conventional optimization techniques, i.e., the steepest descent method and its variants can be computationally expensive. Therefore, most wireless geolocation approaches proposed in recent years pursue, instead, an ad hoc approach, which performs geolocation estimation based on TOA measurements. The computation complexity is then kept at a manageable level. A possible relation between the optimum MLE approach and the popular TOA methods has not been reported in the existing literature and is the theme of this section.

Recall the *Fisher information matrix* for $\underline{\theta}$ is

$$\mathbf{J}_{\underline{\theta}} = \mathbf{H} \cdot \mathbf{J}_{\underline{\tau}} \cdot \mathbf{H}^*,$$

which are applied to both MLE and TOA positioning. This indicates if $(\mathbf{J}_{\underline{\tau}})_{TOA} = (\mathbf{J}_{\underline{\tau}})_{MLE}$, i.e., $(\lambda_b)_{TOA} = (\lambda_b)_{MLE}$ (see Eq. (15) for the definition of λ_b), is satisfied, the CRLB of an MLE receiver is equivalent to that of TOA positioning. In fact, this equivalence can be established, if τ_b's are estimated at the matched filter output with high SNR, which is a well-known results in radar theory [9]. An immediate significance of this result is that the (presumably) complicated MLE solution is now decomposed into the two simple and practical steps: time delay estimation and positioning with τ_b's. Hence the existing TOA based methods can serve as critical components of an overall geolocation algorithm for an MLE solution.

We can establish the relation between SS based and TOA positioning with a similar technique.

IV. Generalization of the CRLB for parameter estimation with unwanted parameters

If we examine the proof of Eq. (18) carefully, we may notice a same consequence, that the CRLB of the MS position ignore the existence of NLOS signals, can hold by replacing Eqs. (3)–(6) with some less specific conditions.

We now present a generalized conclusion of the CRLB for parameter estimation with unwanted parameters. Let $\mathcal{B} = \{1, 2, \cdots, B\}$ be the set of indices of the B receivers. Given the observations $\{r_b(t, \tau_b)\}_{b=1}^B$, where τ_b's are not necessary to be time delays, we are interested in estimating parameters $\underline{u} =$

$(u_1, u_2, \cdots, u_N)$. However, some unwanted parameters $\underline{v} = (v_1, v_2, \cdots, v_K)$ inevitably exist. $\underline{u}$, $\underline{v}$ and $\underline{\tau} = (\tau_1, \tau_2, \cdots, \tau_B)$ are related as

$$\tau_b = \begin{cases} g_b(\underline{u}, \underline{v}), & \text{for } b \in \mathcal{M} \\ g_b(\underline{u}), & \text{for } b \in \mathcal{L}, \end{cases} \tag{21}$$

where $\mathcal{M} = \{k_1, k_2, \cdots, k_M\}$ is a subset of $\mathcal{B}$, the set of the indices of the receivers which receives signals containing unwanted parameters. $\mathcal{L}$ is clear to be $\mathcal{B} \setminus \mathcal{M}$. We assume $\mathcal{M} = \{1, 2, \cdots, M\}$ without loss of generality. Define $\underline{\theta} = (\underline{u}, \underline{v})$. For the joint probability density function of $\{r_b(t, \tau_b)\}_{b=1}^{B}$, denoted as $f_{\underline{\theta}}(\underline{r})$, we make the following assumption

$$f_{\underline{\theta}}(\underline{r}) = f_1(\tau_b(\underline{u}), r_b,\ b \in \mathcal{L}) \cdot f_2(\tau_b(\underline{u}, \underline{v}), r_b,\ b \in \mathcal{M}), \tag{22}$$

which is true in many practical problems. With a similar argument as before, we have the Fisher information matrix for $\underline{\theta}$ as

$$\mathbf{J}_{\underline{\theta}} = \mathbf{H} \cdot \mathbf{J}_{\underline{\tau}} \cdot \mathbf{H}^*,$$

where $\mathbf{J}_{\underline{\tau}}$ is also a Fisher information and

$$\mathbf{H} = \left(\begin{array}{ccc|cc} \frac{\partial \tau_1}{\partial u_1} & \cdots & \frac{\partial \tau_M}{\partial u_1} & \cdots & \frac{\partial \tau_B}{\partial u_1} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \frac{\partial \tau_1}{\partial u_N} & \cdots & \frac{\partial \tau_M}{\partial u_N} & \cdots & \frac{\partial \tau_B}{\partial u_N} \\ \hline \frac{\partial \tau_1}{\partial v_1} & \cdots & \frac{\partial \tau_M}{\partial v_1} & \cdots & \frac{\partial \tau_B}{\partial v_1} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \frac{\partial \tau_1}{\partial v_K} & \cdots & \frac{\partial \tau_M}{\partial v_K} & \cdots & \frac{\partial \tau_B}{\partial v_K} \end{array} \right). \tag{23}$$

$\mathbf{J}_{\underline{\tau}}$ is evaluated as

$$\mathbf{J}_{\underline{\tau}} = \begin{pmatrix} \mathbf{Q}_1(\mathcal{M}) & \mathbf{0} \\ \mathbf{0} & \mathbf{Q}_2(\mathcal{L}) \end{pmatrix}, \tag{24}$$

where $\mathbf{Q}_1(\mathcal{L})$ and $\mathbf{Q}_2(\mathcal{M})$ are not necessary to be diagonal matrices as in Eq. (14). $\mathcal{L}$ and $\mathcal{M}$ in the parentheses denote the corresponding quantities are associated to LOS and NLOS signals respectively. Substitute Eq. (21) into (23), it is straightforward to obtain

$$\mathbf{H} = \begin{pmatrix} \mathbf{H}_1(\mathcal{M}) & \mathbf{H}_2(\mathcal{L}) \\ \mathbf{H}_3(\mathcal{M}) & \mathbf{0} \end{pmatrix}. \tag{25}$$

With Eqs. (24) and (25), it is straightforward to obtain

$$\mathbf{J}_{\underline{\theta}} = \begin{pmatrix} \mathbf{H}_1\mathbf{Q}_1\mathbf{H}_1^H(\mathcal{M}) + \mathbf{H}_2\mathbf{Q}_2\mathbf{H}_2^H(\mathcal{L}) & \mathbf{H}_1\mathbf{Q}_1\mathbf{H}_3^H(\mathcal{M}) \\ \mathbf{H}_3\mathbf{Q}_1\mathbf{H}_1^H(\mathcal{M}) & \mathbf{H}_3\mathbf{Q}_1\mathbf{H}_3^H(\mathcal{M}) \end{pmatrix}. \tag{26}$$

We claim that if (1) det $\left(\mathbf{H}_3\mathbf{Q}_1\mathbf{H}_3^H\right) \neq 0$; (2) there exists an $N \times K$ matrix $\mathbf{T}$ such that $\mathbf{H}_1(\mathcal{M}) = \mathbf{T} \cdot \mathbf{H}_3(\mathcal{M})$, the CRLB for $\underline{u}$ are

$$E_{\underline{\theta}}([u_n - \hat{u}_n]^2) \geq \left(\mathbf{J}_{\underline{\theta}}^{-1}\right)_{nn} = \left(\left[\mathbf{H}_2\mathbf{Q}_2\mathbf{H}_2^H(\mathcal{L})\right]^{-1}\right)_{nn}, \quad \text{for } 1 \leq n \leq N, \tag{27}$$

where $\hat{u}_n$ is the estimate of u_n. The proof uses the exact trick in derivation of Eq. (18). Thus the contribution of signals from receivers in $\mathcal{M}$ are completely ignored. We now see that the CRLB for NLOS geolocation in Section II is nothing but a special case of this generalized layout.

V. Conclusions

In this paper, we present a unified approach for the CRLB for geolocation in the NLOS environment. We explore its physical interpretation for a better understanding of NLOS geolocation procedure, which can provide a useful guidance in devising a specific algorithm. We generalize the formulation of the CRLB for NLOS geolocation to a broader settings of parameter estimation with unwanted parameters, where signals in $\mathcal{M}$ do not influence the lower bound.

Appendix: Derivation of Eq. (18)

It suffices to show

$$\left[\mathbf{J}_{\underline{\theta}}^{-1}\right]_{11} = \left[\left(\mathbf{H}_2\Sigma_2\mathbf{H}_2^H(\mathcal{L})\right)^{-1}\right]_{11}, \tag{28}$$

which can be done in the following four steps:

1. It is clear that

$$\left[\mathbf{J}_{\underline{\theta}}^{-1}\right]_{11} = \frac{\left|\widetilde{\mathbf{J}}_{\underline{\theta}(1,1)}\right|}{\left|\mathbf{J}_{\underline{\theta}}\right|}, \tag{29}$$

where $\widetilde{\mathbf{J}}_{\underline{\theta}(1,1)}$ is an $(M+1) \times (M+1)$ matrix obtained by deleting the first row and the first column of $\mathbf{J}_{\underline{\theta}}$, and $|\cdot|$ is to take determinant of a matrix.

2.

$$\begin{aligned} \left|\mathbf{J}_{\underline{\theta}}\right| &= \begin{vmatrix} \mathbf{H}_1\mathbf{\Sigma}_1\mathbf{H}_1^H + \mathbf{H}_2\mathbf{\Sigma}_2\mathbf{H}_2^H & \frac{1}{c}\mathbf{H}_1\mathbf{\Sigma}_1 \\ \frac{1}{c}\mathbf{\Sigma}_1\mathbf{H}_1^H & \frac{1}{c^2}\mathbf{\Sigma}_1 \end{vmatrix} \\ &= \begin{vmatrix} \mathbf{H}_2\mathbf{\Sigma}_2\mathbf{H}_2^H & \mathbf{0} \\ \frac{1}{c}\mathbf{\Sigma}_1\mathbf{H}_1^H & \frac{1}{c^2}\mathbf{\Sigma}_1 \end{vmatrix} \\ &= \left|\mathbf{H}_2\mathbf{\Sigma}_2\mathbf{H}_2^H\right| \cdot \left|\frac{1}{c^2}\mathbf{\Sigma}_1\right|. \end{aligned} \tag{30}$$

3.

$$\begin{aligned} &\left|\widetilde{\mathbf{J}}_{\underline{\theta}(1,1)}\right| \\ &= \begin{vmatrix} \widetilde{\mathbf{H}}_{1(1,0)}\mathbf{\Sigma}_1\widetilde{\mathbf{H}}_{1(1,0)}^H + \widetilde{\mathbf{H}}_{2(1,0)}\mathbf{\Sigma}_2\widetilde{\mathbf{H}}_{2(1,0)}^H & \frac{1}{c}\widetilde{\mathbf{H}}_{1(1,0)}\mathbf{\Sigma}_1 \\ \frac{1}{c}\mathbf{\Sigma}_1\widetilde{\mathbf{H}}_{1(1,0)}^H & \frac{1}{c^2}\mathbf{\Sigma}_1 \end{vmatrix} \\ &= \begin{vmatrix} \widetilde{\mathbf{H}}_{2(1,0)}\mathbf{\Sigma}_2\widetilde{\mathbf{H}}_{2(1,0)}^H & \mathbf{0} \\ \frac{1}{c}\mathbf{\Sigma}_1\widetilde{\mathbf{H}}_{1(1,0)}^H & \frac{1}{c^2}\mathbf{\Sigma}_1 \end{vmatrix} \\ &= \left|\widetilde{\mathbf{H}}_{2(1,0)}\mathbf{\Sigma}_2\widetilde{\mathbf{H}}_{2(1,0)}^H\right| \cdot \left|\frac{1}{c^2}\mathbf{\Sigma}_1\right| \end{aligned} \tag{31}$$

where $\widetilde{\mathbf{H}}_{1(1,0)}$ and $\widetilde{\mathbf{H}}_{2(1,0)}$ are obtained by deleting the first row of $\mathbf{H}_1$ and $\mathbf{H}_2$, respectively.

4. Substituting Eqs. (30) and (31) into Eq. (29), we have

$$\begin{aligned} \text{CRLB} = \left[\mathbf{J}_{\underline{\theta}}^{-1}\right]_{11} &= \frac{\left|\widetilde{\mathbf{H}}_{2(1,0)}\mathbf{\Sigma}_2\widetilde{\mathbf{H}}_{2(1,0)}^H\right|}{\left|\mathbf{H}_2\mathbf{\Sigma}_2\mathbf{H}_2^H\right|} \\ &= \left[\left(\mathbf{H}_2\mathbf{\Sigma}_2\mathbf{H}_2^H\right)^{-1}\right]_{11}. \end{aligned}$$

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