CRAMÉR-RAO LOWER BOUND FOR GEOLOCATION IN NON-LINE-OF-SIGHT ENVIRONMENT

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ABSTRACT

Geolocation in an non-line-of-sight (NLOS) environment is an important issue in wireless communications. Several empirical approaches have been proposed in recent years. However, two fundamental questions still remain unanswered: what is the achievable geolocation accuracy in the NLOS environment and how to attain it. In this paper, we derive a closed-form expression of the Cramér-Rao Lower bound (CRLB) for NLOS geolocation, which we believe is new and answers the first question. Its physical interpretation turns out to be very helpful to better understanding of NLOS geolocation mechanism. We discuss some numerical examples based on simulation experiments.

1. INTRODUCTION

Geolocation in an non-line-of-sight (NLOS) environment is an important issue in mobile communications, and is receiving a considerable attention in recent years. Several empirical methods [1]–[5] have been proposed to mitigate NLOS effects in geolocation. However, two fundamental questions are still not clear: what is the best achievable geolocation accuracy in the NLOS environment and how to improve it. This paper is concentrated with deriving an explicit formula of the Cramér-Rao Lower bound (CRLB) for NLOS geolocation. The CRLB is well-known as a tight lower bound for the error variance of an unbiased estimate of some unknown parameter [6, 7]. We consider a situation in which mobile station (MS) positioning is based on base station (BS) measurements of the signal transmitted by the MS. A new closed-form expression is obtained, which will provide an insightful direction towards construction of NLOS geolocation algorithms. We also discuss a few numerical examples.

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The rest of the paper is organized as follows. In Section 2, the derivation of the CRLB for NLOS geolocation is outlined. Its physical interpretations is provided in Section 3. Section 4 presents some numerical examples. We state a brief conclusion in Section 5.

2. CRLB FOR NLOS GEOLOCATION

The Cramér-Rao inequality gives a lower bound for the error variance of any unbiased estimate of some unknown parameter [6, 7]. Let $\hat{\underline{\theta}}$ be an estimate of the vector of parameters $\underline{\theta}$. Denote $E_{\underline{\theta}}\{\cdot\}$ as the expected value conditioned on $\underline{\theta}$. The CRLB is expressed as

$$\mathrm{Cov}_{\underline{\theta}}(\hat{\underline{\theta}}) \geq \mathbf{I}_{\underline{\theta}}^{-1}, \tag{1}$$

where $\mathrm{Cov}_{\underline{\theta}}(\hat{\underline{\theta}}) \equiv E_{\underline{\theta}}\left\{(\hat{\underline{\theta}}-\underline{\theta})(\hat{\underline{\theta}}-\underline{\theta})^T\right\}$, and $\mathbf{I}_{\underline{\theta}}$ is the $n \times n$ *Fisher information matrix*, whose (j,l) element is defined as

$$\left(\mathbf{I}_{\underline{\theta}}\right)_{j,l} = E_{\underline{\theta}}\left\{\frac{\partial}{\partial\theta_j}\log f_{\underline{\theta}}(\underline{r}) \cdot \frac{\partial}{\partial\theta_l}\log f_{\underline{\theta}}(\underline{r})\right\}, \quad \text{for } j,l = 1,2,\cdots,n, \tag{2}$$

where $f_{\underline{\theta}}(\underline{r})$ is the probability density function of measured data $\underline{r}$ conditioned on the unknown parameters $\underline{\theta}$.

Let $s(t)$ be the signal transmitted from the MS of our interest, which is located at $\vec{p} = (x,y)$. Let $\mathcal{B} = \{1,2,\cdots,B\}$ be the set of indices of the B base stations, which are at $\{\vec{p}_b = (x_b,y_b)\}_{b=1}^{B}$. Denote the set of indices of the BS's which receive NLOS signals as $\mathcal{M} = \{k_1,k_2,\cdots,k_M\}$. Thus, the complement $\mathcal{B} \setminus \mathcal{M} \equiv \mathcal{L}$ is the set of LOS base stations, and its cardinality is $|\mathcal{L}| = B - M$. Let τ_b be the time delay of the signal at base station b (BS_b). The received signal at BS_b is

$$r_b(t) = A_b s(t-\tau_b) + n_b(t), \quad \text{for } b \in \mathcal{B}, \tag{3}$$

with

$$\tau_b = \frac{1}{c}\left\{\sqrt{(x_b - x)^2 + (y_b - y)^2} + D_b\right\}, \quad \text{for } b \in \mathcal{B}, \tag{4}$$

where $\{D_b, b \in \mathcal{M}\}$ represents the unknown additional path length due to NLOS propagation while $D_b = 0$ if $b \in \mathcal{L}$. $c = 3 \times 10^8\ m/s$ is the speed of light and $n_b(t)$'s are complex-valued white Gaussian noise processes with spectral density $N_0/2$. Define

$$\underline{\theta} = (\theta_1, \theta_2, \cdots, \theta_{M+2})^T = (x, y, D_{k_1}, \cdots, D_{k_M})^T.$$

The joint probability density function of $\{r_b(t)\}_{b=1}^{B}$ is

$$\begin{aligned} f_{\underline{\theta}}(\underline{r}) &\equiv f(\underline{r}|x, y, D_{k_1}, D_{k_2}, \cdots, D_{k_M}) \\ &\propto \prod_{b=1}^{B} \exp\left\{-\frac{1}{N_0}\int |r_b(t) - A_b s(t-\tau_b)|^2\, dt\right\} \end{aligned} \tag{5}$$

Substituting Eq. (5) into Eq. (2), it is straightforward to obtain

$$\begin{aligned}
[\mathbf{I}_{\underline{\theta}}]_{x,x} &= \alpha \sum_{b=1}^{B} R_b \frac{(x-x_b)^2}{(x-x_b)^2 + (y-y_b)^2}, \\
[\mathbf{I}_{\underline{\theta}}]_{y,y} &= \alpha \sum_{b=1}^{B} R_b \frac{(y-y_b)^2}{(x-x_b)^2 + (y-y_b)^2}, \\
[\mathbf{I}_{\underline{\theta}}]_{x,y} &= [\mathbf{I}_{\underline{\theta}}]_{y,x}, \\
&= \alpha \sum_{b=1}^{B} R_b \frac{(x-x_b)(y-y_b)}{(x-x_b)^2 + (y-y_b)^2}, \\
[\mathbf{I}_{\underline{\theta}}]_{x,D_{k_m}} &= [\mathbf{I}_{\underline{\theta}}]_{D_{k_m},x}, \\
&= \alpha R_{k_m} \frac{x - x_{k_m}}{\sqrt{(x-x_{k_m})^2 + (y-y_{k_m})^2}}, \\
&\quad \text{for } m = 1, 2, \cdots, M, \\
[\mathbf{I}_{\underline{\theta}}]_{y,D_{k_m}} &= [\mathbf{I}_{\underline{\theta}}]_{D_{k_m},y}, \\
&= \alpha R_{k_m} \frac{y - y_{k_m}}{\sqrt{(x-x_{k_m})^2 + (y-y_{k_m})^2}}, \\
&\quad \text{for } m = 1, 2, \cdots, M, \\
[\mathbf{I}_{\underline{\theta}}]_{D_{k_m},D_{k_m}} &= \alpha R_{k_m}, \\
&\quad \text{for } m = 1, 2, \cdots, M, \\
[\mathbf{I}_{\underline{\theta}}]_{j,l} &= 0, \quad \text{otherwise},
\end{aligned} \tag{6}$$

where R_b is the signal-to-noise ratio (SNR) of the received signal at BS_b, i.e.,

$$R_b = \frac{\int |A_b s(t)|^2 dt}{N_0},$$

and

$$\alpha = \frac{8\pi^2\beta^2}{c^2}$$

is proportional to β^2. β is the effective bandwidth of the signal waveform, defined as

$$\beta^2 = \int f^2 |S(f)|^2 df,$$

where $S(f)$ is the Fourier transform of $s(t)$. Throughout this paper, we assume the following normalization condition:

$$\int |s(t)|^2 dt = 1.$$

Now we derive the explicit expression of the CRLB for NLOS geolocation. Substituting Eq. (6) into Eq. (1), we have

$$E_{\underline{\theta}}([x - \hat{x}]^2) \geq \left[\mathbf{I}_{\underline{\theta}}^{-1}\right]_{11} = \frac{\det\left(\widetilde{\mathbf{I}}_{\underline{\theta}(1,1)}\right)}{\det\left(\mathbf{I}_{\underline{\theta}}\right)},$$

where $\widetilde{\mathbf{I}}_{\underline{\theta}(1,1)}$ is an $(M+1) \times (M+1)$ matrix obtained by deleting the first row and the first column of $\mathbf{I}_{\underline{\theta}}$. We note both $\mathbf{I}_{\underline{\theta}}$ and $\widetilde{\mathbf{I}}_{\underline{\theta}(1,1)}$ can be decomposed into matrices of a special structure

$$\mathbf{F} = \begin{pmatrix} a & f_1 & f_2 & f_3 & \cdots & f_K \\ e_1 & d_1 & 0 & 0 & \cdots & 0 \\ e_2 & 0 & d_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ e_K & 0 & 0 & \cdots & 0 & d_K \end{pmatrix}.$$

The computation complexity can be greatly reduced if we take advantage of the following relation

$$\det \mathbf{F} = \prod_{k=1}^{K} d_k \cdot \left(a - \sum_{k=1}^{K} \frac{f_k e_k}{d_k}\right).$$

Define angle ϕ_b by

$$\tan^{-1}\phi_b = \frac{x - x_b}{y - y_b}, \quad \text{for } b \in \mathcal{B}. \tag{7}$$

After some algebraic manipulation, it can be shown that the mean squared Euclidean distance between the true MS position $\vec{p} = (x, y)$ and its estimate $\widehat{\vec{p}} = (\hat{x}, \hat{y})$ is lower bounded by

$$\begin{aligned} E_{\underline{\theta}}\left([\vec{p} - \widehat{\vec{p}}]^2\right) &= E_{\underline{\theta}}([x - \hat{x}]^2) + E_{\underline{\theta}}([y - \hat{y}]^2) \\ &\geq \frac{\sum_{b \in \mathcal{L}} R_b}{\alpha \sum \sum_{b_1, b_2 \in \mathcal{L}} R_{b_1} R_{b_2} \sin^2(\phi_{b_1} - \phi_{b_2})}. \end{aligned} \tag{8}$$

If we are interested in estimating the path length D_b for the received signal at base station $b = k_m \in \mathcal{M}$, we can obtain the lower bound of the estimate error following a similar procedure:

$$E_{\underline{\theta}}([D_{k_m} - \widehat{D}_{k_m}]^2) \geq \left[\mathbf{I}_{\underline{\theta}}^{-1}\right]_{m+2,m+2}$$
$$= \frac{1}{\alpha R_{k_m}} \frac{\sum\sum_{b_1,b_2 \in \mathcal{L}\cup\{k_m\}} R_{b_1} R_{b_2} \sin^2(\phi_{b_1} - \phi_{b_2})}{\sum\sum_{b_1,b_2 \in \mathcal{L}} R_{b_1} R_{b_2} \sin^2(\phi_{b_1} - \phi_{b_2})},$$
$$\geq \frac{1}{\alpha R_{k_m}}, \qquad (9)$$

for $m = 1, 2, \cdots, M$.

3. PHYSICAL INTERPRETATION

Now we are in a position to provide a physical interpretation of Eqs. (8) and (9).

First, it is interesting to note that the minimum mean square error of an estimate of $\vec{p} = (x, y)$ depends only on the signals obtained by LOS BS's and NLOS singnals are totally ignored. This implies that if we try to estimate a MS position in an NLOS environment, the performance should be same for the algorithm of distinguishing NLOS signals and rejecting them as that of actually estimating the NLOS induced path length D_{k_m}'s. The former usually requires less computation load. If we need to estimate a D_{k_m} for other purpose, such as channel estimation, we can observe similar conclusion: the other $(M-1)$ NLOS BS's do not contribute to improving the estimate accuracy of $\widehat{D}_{k_m}$. Only the signal received by $(B-M)$ LOS BS's and the NLOS signal at this specific BS_{k_m} matter.

Secondly, the geolocation accuracy of $\vec{p}$ expressed in the MSE (mean-squared error) is inversely proportional to the square of effective bandwidth, β^2, as previously known in radar estimation (see e.g., [8]).

Thirdly, the geolocation accuracy also depends on the geometric relation among the LOS BS's, through sine functions of the angle differences $(\phi_{b_1} - \phi_{b_2})$, $b_1, b_2 \in \mathcal{L}$, seen by the MS of our interest. It becomes infinite large when all LOS BS's and the MS happen to lie on a straight line, i.e.,

$$E_{\underline{\theta}}\left([\vec{p} - \widehat{\vec{p}}]^2\right) \longrightarrow +\infty, \text{ as } \phi_{b_1} - \phi_{b_2} = 0, \pi, \text{ for } b_1, b_2 \in \mathcal{L}.$$

Fortunately, such is rarely the case in reality.

Finally, we discuss the relation between the CRLB and the signal-to-noise ratio R_k, $k \in \mathcal{L}$. We rewrite Eq. (8) as

$$\text{CRLB} = \frac{1}{\alpha C}\left\{1 + \frac{AC - G}{G + CR_k}\right\}, \qquad (10)$$

where

$$A = \sum_{b \neq k} R_b \geq 0,$$
$$G = \sum \sum_{b_1, b_2 \neq k} R_{b_1} R_{b_2} \sin^2(\phi_{b_1} - \phi_{b_2}) \geq 0,$$
$$C = \sum_{b \neq k} R_b \sin^2(\phi_b - \phi_k) \geq 0. \qquad (11)$$

It can be shown that

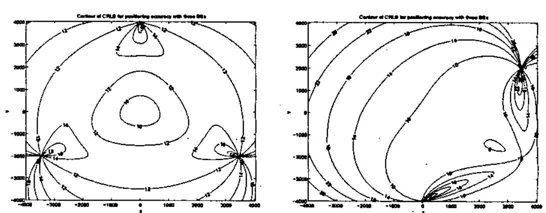
$$(AC - G) \geq 0, \text{ with}$$
$$(AC - G) = 0, \text{ iff. } \phi_b - \phi_k = 0, \pi, \; b \neq k. \qquad (12)$$

See Appendix for the detailed derivation. Thus, the geolocation accuracy should improve as the R_k increases if $(AC - G) > 0$, which corresponds to the case that all LOS BS's and the MS are not on a line.

4. NUMERICAL EXAMPLES

We present some numerical examples in this section. Since NLOS BS's do not contribute to enhancing the achievable performance, we assume the BS's considered in this section receive LOS signals without loss of generality. In simulation, all BS's are located along a circle with radius of $4000m$. The MS can move freely on the two-dimensional plane and transmit a CDMA signal.

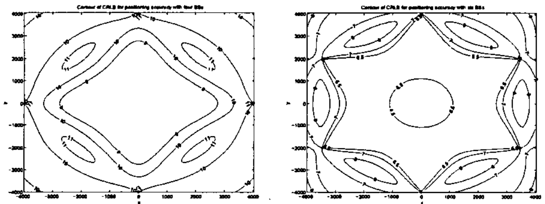
The contours of square root of the CRLB with different BS layouts are shown in Figs. 1 and 2. The SNR at each BS is 3dB when MS is at (0,0). The CDMA signal from the MS is $5Mcps$. The BS locations are denoted as "*" in the plots. As expected, the contour is symmetric if all BS's is evenly distributed on the circle. The positioning error tends to be large along the line between two BS's.



(a) (b)

Fig. 1. Contour of $\sqrt{\text{CRLB}}$ (in meter) with three BS's. BS positions are denoted by "*". SNR is 3dB at each BS when MS is at (0,0). (a) The three BS's are evenly distributed on the circle. (b) The BS's are unevenly distributed.

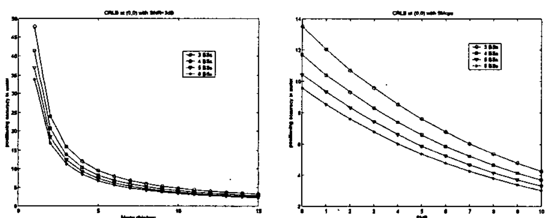
Figure 3 shows that a better geolocation accuracy is achieved with higher SNR, higher chip rate and more BS's involved. The performance may not significantly improve, however,



(a) (b)

Fig. 2. Contour of $\sqrt{\text{CRLB}}$ (in meter) with four BS's in (a) and six BS's in (b). BS positions are denoted by "*". SNR is 3dB at each BS when MS is at (0,0).

beyond some threshold, say, in Fig. 3(a), about $5Mcps$. In most cases here, the lower bound is well below $50m$.



(a) (b)

Fig. 3. $\sqrt{\text{CRLB}}$ with MS at (0,0) in different scenarios. (a) Chip rate vs. position accuracy with SNR=3dB; (b) SNR vs. position accuracy with $5Mcps$.

5. CONCLUSION

In this paper, we present a new expression for the CRLB for geolocation in the NLOS environment – an achievable geolocation accuracy. We explore its physical interpretation for a better understanding of NLOS geolocation mechanism, which can provide a useful guidance in devising a specific algorithm. Numerical examples are discussed in the end, which shows positioning accuracy of less than $50m$ is available in most cases. One limitation with the derivation here is that we do not impose the condition $\{D_k > 0,\ k \in \mathcal{M}\}$, although the NLOS induced path length is always positive. We currently investigate the CRLB with prior information on some parameters, e.g., $\{D_k > 0,\ k \in \mathcal{M}\}$.

6. REFERENCES

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Appendix: Derivation of Eq. (12)

Let $b_1, b_2, b, k \in \mathcal{L}$ and $\xi_b = \phi_b - \phi_k$, for $b \neq k$. By substituting Eq. (11) into $(AC - G)$, we have

$$
\begin{aligned}
& AC - G \\
&= \sum \sum_{b_1, b_2 \neq k} R_{b_1} R_{b_2} \left(\sin^2 \xi_{b_2} - \sin^2(\xi_{b_1} - \xi_{b_2}) \right) \\
&= \sum_{b \neq k} R_b^2 \sin^2(\xi_b) + 2 \sum \sum_{b_1 < b_2,\ b_1, b_2 \neq k} R_{b_1} R_{b_2} \cdot \\
& (\sin^2 \xi_{b_1} \sin^2 \xi_{b_2} + \sin \xi_{b_1} \sin \xi_{b_2} \cos \xi_{b_1} \cos \xi_{b_2}) \\
& \left(\text{using } \textstyle\sum_{b \neq k} R_b^2 \sin^2(\xi_b) \geq \sum_{b \neq k} R_b^2 \sin^2(\xi_b) \cos^2(\xi_b) \right) \\
& \geq \Bigg(\sum_{b \neq k} R_b^2 \sin^2(\xi_b) \cos^2(\xi_b) + \\
& 2 \sum \sum_{b_1 < b_2,\ b_1, b_2 \neq k} R_{b_1} R_{b_2} \sin \xi_{b_1} \sin \xi_{b_2} \cos \xi_{b_1} \cos \xi_{b_2} \Bigg) \\
& + 2 \sum \sum_{b_1 < b_2,\ b_1, b_2 \neq k} R_{b_1} R_{b_2} \sin^2 \xi_{b_1} \sin^2 \xi_{b_2} \\
&= \left(\sum_{b \neq k} R_b \sin(\xi_b) \cos(\xi_b) \right)^2 + \\
& 2 \sum \sum_{b_1 < b_2,\ b_1, b_2 \neq k} R_{b_1} R_{b_2} \sin^2 \xi_{b_1} \sin^2 \xi_{b_2} \\
& \geq 0.
\end{aligned}
$$

It is clear that $(AC - G) = 0$ if and only if for $\phi_b - \phi_k = 0, \pi,\ b \neq k$. $\Diamond$