On Relation among Time Delay and Signal Strength based Geolocation Methods

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***Abstract*— The time-of-arrival (TOA), time-difference-of-arrival (TDOA) and signal strength (SS) methods have been widely accepted as three principal techniques for positioning a mobile station (MS) in a wireless communication system. To the best of our knowledge, previous studies tend to treat these methods separately, and less analytical results on their relationship have been reported. In this paper, the link between the TOA and TDOA methods is first examined. We provided an analytical explanation for the claim that given a set of BS locations and an MS position, the TOA method should achieve a higher positioning precision than its TDOA counterpart. However, the two positioning methods may attain the same level of accuracy under certain conditions. We then investigate the tradeoff between the accuracy limits of the TOA and SS based methods, which leads to our proposal of a hybrid distance estimation scheme that combines both TOA and SS data.**

I. Introduction

Geolocation, that is, positioning a mobile station (MS) in a wireless communication system, has received a considerable attention with the fast development of mobile communications in recent years. The time-of-arrival (TOA), time-difference-of-arrival (TDOA) and signal strength (SS) based approaches are three major geolocation techniques. Although the rationale for these techniques may largely lie in a triangulation argument, our recent studies [1][2] based on the theory of maximum likelihood estimation (MLE) indeed confirm that these geometric methods serve as critical components of theoretically optimum positioning receivers. Most of the existing studies [3] in the literature tend to treat these techniques separately. However, a better understanding of the connections among the three methods is of both theoretical and practical interest.

In this paper, we report our progress towards this direction. We first explore the link between the TOA and TDOA positioning methods. Given a set of BSs and an MS, we know in principle that the TOA based method should perform better than the TDOA method, because in the former the system is synchronized, whereas in the latter there exists an unknown time offset among the MS and BSs. Here we provide an analytical explanation for the argument. It is also shown that the two positioning methods may achieve the same accuracy under certain conditions. In general, however, we may not find comparable TOA positioning configurations which can approximate a given TDOA system. Our second topic is to pursue the relationship between the accuracy limit for the TOA and SS positioning. By taking advantage of the two methods, we propose a hybrid geolocation scheme that combines TOA and SS data.

The rest of the paper is structured as follows. In Section II, we briefly review the definition of the CRLB and the CRLBs for the TOA and TDOA methods, which will facilitate the development in the remaining sections. We then examine the connection between TOA and TDOA methods in Section III. By comparing the accuracy limits of TOA and SS based distance estimation in Section IV, we devise a new geolocation algorithm in Section V. Section VI concludes this paper.

II. Review

It is well known that the Cramer-Rao Lower Bound (CRLB) sets a lower limit for the variance (or covariance matrix) of any unbiased estimates of an unknown parameter (or unknown parameters) [4][5]. Denote $\hat{\boldsymbol{\theta}}$ as an estimate of the vector of parameters $\boldsymbol{\theta}$. Its covariance matrix is

$$\mathrm{Cov}_{\boldsymbol{\theta}}(\hat{\boldsymbol{\theta}}) = E_{\boldsymbol{\theta}}\left\{(\hat{\boldsymbol{\theta}}-\boldsymbol{\theta})(\hat{\boldsymbol{\theta}}-\boldsymbol{\theta})^T\right\},$$

where $E_{\boldsymbol{\theta}}\{\cdot\}$ stands for an expectation conditioned on $\boldsymbol{\theta}$ and symbol "T" is for matrix transpose. Let $f_{\boldsymbol{\theta}}(\boldsymbol{r})$ be the probability density function (p.d.f.) of observations $\boldsymbol{r}$ conditioned on $\boldsymbol{\theta}$. Its *Fisher information matrix* (FIM) is given by

$$\mathbf{J}_{\boldsymbol{\theta}} = E_{\boldsymbol{\theta}}\left\{\frac{\partial}{\partial\boldsymbol{\theta}}\log f_{\boldsymbol{\theta}}(\boldsymbol{r})\cdot\left(\frac{\partial}{\partial\boldsymbol{\theta}}\log f_{\boldsymbol{\theta}}(\boldsymbol{r})\right)^T\right\}. \quad (1)$$

The CRLB is then expressed as

$$\mathrm{Cov}_{\boldsymbol{\theta}}(\hat{\underline{\theta}}) \geq \mathbf{J}_{\boldsymbol{\theta}}^{-1}. \quad (2)$$

Let $\mathcal{L} = \{1, 2, \cdots, L\}$ be the set of indices of L base stations, whose locations are known at $\{\boldsymbol{p}_b = (x_b, y_b)^T,\ b \in \mathcal{L}\}$. The parameter of our interest is the MS position, denoted by $\boldsymbol{p} = (x, y)^T$. The received signal at base station b (BS_b) is

$$r_b(t) = A_b s(t-\tau_b) + n_b(t), \quad \text{for } b \in \mathcal{L}, \quad (3)$$

[1] This work has been supported, in part, by grants for the New Jersey Center for Wireless Telecommunications (NJCWT) and NTT DoCoMo Inc..

where $s(t)$ is the base-band signal waveform, τ_b and A_b are the time delay and signal amplitude respectively, and $n_b(t)$'s are independent complex-valued white Gaussian noise processes with spectral density $N_0/2$. In a synchronous communication system, which corresponds to TOA positioning, the time delay is

$$\tau_b = \frac{1}{c}\sqrt{(x-x_b)^2 + (y-y_b)^2}, \quad \text{for } b \in \mathcal{L}, \tag{4}$$

where $c = 3 \times 10^8$m/s is the speed of light. The CRLB for an MS estimate is derived as [1][6]

$$\mathbf{J}_{TOA}^{-1} = \left(\mathbf{H}_{TOA} \cdot \mathbf{\Lambda} \cdot \mathbf{H}_{TOA}^T\right)^{-1}, \tag{5}$$

where

$$\mathbf{H}_{TOA} = \frac{1}{c}\begin{pmatrix} \cos\phi_1 & \cos\phi_2 & \cdots & \cos\phi_L \\ \sin\phi_1 & \sin\phi_2 & \cdots & \sin\phi_L \end{pmatrix},$$

and

$$\mathbf{\Lambda} = \text{diag}\left(\lambda_1,\ \lambda_2,\ \cdots, \lambda_L\right).$$

Angle ϕ_b is determined by

$$\phi_b = \tan^{-1}\frac{y-y_b}{x-x_b}, \quad \text{for } b \in \mathcal{L}, \tag{6}$$

which is the geometric angle between the positions of the MS and BS$_b$. The diagonal term of $\mathbf{\Lambda}$ is

$$\lambda_b = 8\pi^2\beta^2 \cdot R_b, \quad \text{for } b \in \mathcal{L}, \tag{7}$$

where R_b is the SNR of the received signal at BS$_b$, i.e.,

$$R_b = \frac{\int |A_b s(t)|^2 dt}{N_0} = \frac{A_b^2}{N_0}. \tag{8}$$

We assume the normalization condition

$$\int |s(t)|^2 dt = 1, \tag{9}$$

for simplicity. Parameter β is the effective bandwidth of the signal waveform $s(t)$, defined as

$$\beta^2 = \int f^2 |S(f)|^2 df, \tag{10}$$

where $S(f)$ is the Fourier transform of $s(t)$. Note that $\mathbf{\Lambda}$ contains the system parameters.

For a non-synchronous system, there is a unknown time offset between the clock at an MS and those at BSs. So the time delay becomes

$$\tau_b = \frac{1}{c}\left(\sqrt{(x-x_b)^2 + (y-y_b)^2} + l_0\right), \quad \text{for } b \in \mathcal{L}, \tag{11}$$

where l_0/c is the time offset. The optimum geolocation scheme for a non-synchronous system was shown to be the TDOA based maximum likelihood estimation (MLE) in [2]. The associated FIM can be expressed in a form similar to that for the TOA method:

$$\mathbf{J}_{TDOA} = \mathbf{H}_{TDOA} \cdot \mathbf{\Psi}^{-1} \cdot \mathbf{H}_{TDOA}^T, \tag{12}$$

where

$$\mathbf{H}_{TDOA} = \frac{1}{c}\begin{pmatrix} \cos\phi_1 & \cos\phi_2 & \cdots & \cos\phi_{L-1} \\ \sin\phi_1 & \sin\phi_2 & \cdots & \sin\phi_{L-1} \end{pmatrix} - \frac{1}{c}\begin{pmatrix} \cos\phi_L \\ \sin\phi_L \end{pmatrix} \cdot \mathbf{1}^T, \tag{13}$$

and

$$\mathbf{\Psi} = \text{diag}\left(\lambda_1,\ \lambda_2,\ \cdots, \lambda_{L-1}\right)^{-1} + \lambda_L^{-1}\mathbf{1} \cdot \mathbf{1}^T, \tag{14}$$

where $\mathbf{1}$ is an (L-1)-dimensional vector with "1" for each entry.

III. Relationship between TOA and TDOA Methods

In this section, we investigate the relationship between the TDOA and TOA methods by comparing their CRLBs.

To facilitate our development, we first prepare the FIMs associated with the two CRLBs in terms of outer product of vectors. By defining unit vector

$$\boldsymbol{h}_b = \begin{pmatrix} \cos\phi_b \\ \sin\phi_b \end{pmatrix},$$

we rewrite the TOA FIM in Eq. (5) as

$$\mathbf{J}_{TOA} = \frac{1}{c^2}\sum_{b\in\mathcal{L}} \lambda_b \boldsymbol{h}_b \boldsymbol{h}_b^T. \tag{15}$$

Since λ_b's are all positive, we define weight coefficient w_b as

$$w_b = \frac{\lambda_b}{\lambda},$$

where

$$\lambda = \sum_{b\in\mathcal{L}} \lambda_b.$$

Thus, Eq. (15) becomes

$$\mathbf{J}_{TOA} = \frac{\lambda}{c^2}\sum_{b\in\mathcal{L}} w_b \boldsymbol{h}_b \boldsymbol{h}_b^T. \tag{16}$$

Similarly, the TDOA FIM of Eq. (12) is derived as

$$\mathbf{J}_{TDOA} = \frac{\lambda}{c^2}\left(\sum_{b\in\mathcal{L}} w_b \boldsymbol{h}_b \boldsymbol{h}_b^T - (\sum_{b\in\mathcal{L}} w_b \boldsymbol{h}_b)(\sum_{b\in\mathcal{L}} w_b \boldsymbol{h}_b)^T\right). \tag{17}$$

Define random vector $\boldsymbol{h}$ that takes values of

$$\boldsymbol{h}_1, \boldsymbol{h}_2, \cdots, \boldsymbol{h}_L$$

with probabilities

$$w_1, w_2, \cdots, w_L,$$

and its weighted average

$$\bar{\boldsymbol{h}} = \sum_{b=1}^{L} w_b \boldsymbol{h}_b.$$

We can express $\mathbf{J}_{TOA}$ and $\mathbf{J}_{TDOA}$ in terms of the second moment and covariance of $\boldsymbol{h}$, respectively, i.e.,

$$\mathbf{J}_{TOA} = \frac{\lambda}{c^2} \cdot E\left\{\boldsymbol{h} \cdot \boldsymbol{h}^T\right\}, \tag{18}$$

and

$$\mathbf{J}_{TDOA} = \frac{\lambda}{c^2} \cdot E\left\{(\boldsymbol{h} - \bar{\boldsymbol{h}}) \cdot (\boldsymbol{h} - \bar{\boldsymbol{h}})^T\right\}. \tag{19}$$

By utilizing Eqs. (16) and (17), we immediately see

$$\mathbf{J}_{TOA} - \mathbf{J}_{TDOA} = \frac{\lambda}{c^2} \cdot \bar{\boldsymbol{h}} \cdot \bar{\boldsymbol{h}}^T \geq 0. \tag{20}$$

Hence,

$$\mathbf{J}_{TDOA}^{-1} \geq \mathbf{J}_{TOA}^{-1}. \tag{21}$$

This inequality confirms the long-held argument that as far as we use the same set of time delay estimates $\{\hat{\boldsymbol{\tau}}_b,\ b \in \mathcal{L}\}$, TDOA positioning cannot perform better than the TOA method because of the extra unknown, i.e., the time-offset l_0 in the non-synchronous system. The amount of degradation is given in Eq. (20). Thus, we see that weighted average $\bar{\boldsymbol{h}}$ is a crucial quantity in determining the degradation amount. Moreover, the sufficient and necessary condition for the equality to hold in Eq. (21) is

$$\mathbf{J}_{TDOA}^{-1} = \mathbf{J}_{TOA}^{-1} \quad \Leftrightarrow \quad \bar{\boldsymbol{h}} = 0. \tag{22}$$

That is, the TOA and TDOA positioning methods can attain the same accuracy if and only if when $\bar{\boldsymbol{h}}$ is exactly zero, which may be viewed as a kind of symmetry condition among the configuration among the BSs and the MS. This symmetry can annul the accuracy degradation of the TDOA method with respect to the TOA accuracy. Here is a simple example for $\bar{\boldsymbol{h}} = 0$. Consider that L BSs are distributed evenly around the circle with the center at the mobile's location. We then have

$$w_1 = w_2 = \cdots = w_L,$$

and

$$\sum_{b \in \mathcal{L}} \boldsymbol{h}_b = 0.$$

Thus, $\bar{\boldsymbol{h}} = 0$.

The relation of Eq. (21) implies that $\mathbf{J}_{TDOA}^{-1}$, the CRLB of the TDOA method, is lower bounded by $\mathbf{J}_{TOA}^{-1}$. Hence, a closely related question is raised: does there exist an upper bound for $\mathbf{J}_{TDOA}^{-1}$ determined by some TOA configuration? To be more specific, denote $\mathbf{J}_{TOA}(l)$ the TOA FIM associated with l BSs of (BS_1, BS_2, $\cdots$, BS_l), which is the subset of $\mathcal{L}$. Can we find an $0 < l < L$ such that

$$\mathbf{J}_{TDOA}^{-1} \leq \mathbf{J}_{TOA}(l)^{-1} \ \textbf{?}$$

We pose this question, since it is conceivable that use of fewer BSs should reduce the positioning accuracy. Along with the lower bound of Eq. (21), the plausible relation

$$\mathbf{J}_{TOA}^{-1}(L) \leq \mathbf{J}_{TDOA}^{-1} \leq \mathbf{J}_{TOA}^{-1}(l)$$

would possibly allow to approximate the performance of a given TDOA scheme by two related TOA solutions. The conjectured upper bound is equivalent to requiring

$$\sum_{b=l+1}^{L} w_b \boldsymbol{h}_b \boldsymbol{h}_b^T \geq (\sum_{b \in \mathcal{L}} w_b \boldsymbol{h}_b)(\sum_{b \in \mathcal{L}} w_b \boldsymbol{h}_b)^T. \tag{23}$$

However, the answer depends on the specific configuration of $\{\boldsymbol{h}_b,\ b \in \mathcal{L}\}$ and $\{w_b,\ b \in \mathcal{L}\}$. We provide two examples in Appendix 1, where such an l exists in one example and $l = 0$ in the other.

IV. Relation between TOA and SS positioning methods

Information regarding $\boldsymbol{p}$, the mobile position, is contained in both the arrival times in the received waveforms and their amplitudes. The TOA and TDOA methods are based on the former type of data, whereas the signal-strength method uses the second type of data. The advantage of the SS method is in two folds. On the one hand, we may utilize some simple devices for measuring the receiving energy. The other is that its straightforward formulation can capture the major factors imposed by a harsh mobile environment, that is, the path loss factor $\epsilon > 2$ ($\epsilon = 2$ for the free space), Rayleigh fading due to multipath by the local scatterers, and log-normal shadowing caused by the obstacles in the propagation paths. However, the major limitation for the SS positioning method is its poor precision when locating an MS within a wide region. We elaborate on this claim by examining the achievable accuracy of the SS method along with that of the TOA method.

Consider a one-dimensional case like in a radar ranging problem. Our task is to estimate the distance d between an MS and one BS based on the SS or TOA estimates.

For the SS method, we use the time averaged of SS data which are formulated as

$$e = z + w,$$

where

$$z = -10 \cdot \epsilon \cdot \log d,$$

w is a Gaussian variable $\mathcal{N}(0, \eta^2)$ representing log-normal shadowing, and ϵ is the path loss factor. Hence, the p.d.f. of e conditioned on d is

$$f_d(e) \propto \exp\left\{ -\frac{1}{2\eta^2} \left(e + 10 \cdot \epsilon \cdot \log d\right)^2 \right\}. \tag{24}$$

It is straightforward to show that the corresponding CRLB is

$$(\mathbf{J}_d)_{SS}^{-1} = \left(\frac{\ln 10}{10}\right)^2 \cdot \frac{\eta^2}{\epsilon^2} \cdot d^2, \tag{25}$$

or

$$\sqrt{\mathrm{var}(\hat{d})} \geq \frac{\ln 10}{10} \cdot \frac{\eta}{\epsilon} \cdot d. \tag{26}$$

Note that the accuracy of the above expression is proportional to d. In other words, in order to maintain the estimation error of less than δd, the MS has to be within the range of

$$r_0 = \frac{10}{\ln 10} \cdot \frac{\epsilon}{\eta} \cdot \delta d \tag{27}$$

from the specific BS's location. For typical numbers $\epsilon = 4$ and $\eta = 8$ pertaining to outdoor geolocation, the accuracy of Eq. (26) is roughly $0.5d$. Thus, to secure the accuracy of 100m, the maximum distance between the MS and the BS is 200m.

We should note in Eq. (26) that ϵ and η^2 are completely determined by the characteristics of a communication channel. So there is little we can do to control or improve these factors and the resulting positioning accuracy.

For TOA positioning, the case is different. If the time delay (i.e., TOA) of the received signal in Eq. (3) is estimated at a matched filter output, the delay estimate can be formulated as [11]

$$\hat{\tau} = \tau + \xi,$$

where ξ is a Gaussian random variable $\mathcal{N}(0, \sigma^2)$ with

$$\sigma^2 = \frac{1}{8\pi^2 \beta^2 R},$$

and R is the SNR. The associated CRLB is given by

$$(\mathbf{J}_d)^{-1}_{TOA} = \frac{c^2}{8\pi^2\beta^2 \cdot R}, \tag{28}$$

or equivalently,

$$\sqrt{\mathrm{var}(\hat{d})} \geq \frac{c}{2\sqrt{2}\pi} \cdot \frac{1}{\beta} \cdot \frac{1}{\sqrt{R}}. \tag{29}$$

Consider a CDMA signal with bandwidth W. By using the relation between W and the effective bandwidth β [1]

$$\beta = \frac{W}{\sqrt{12}},$$

we obtain

$$\sqrt{\mathrm{var}(\hat{d})} \geq \frac{\sqrt{3}c}{\sqrt{2}\pi} \cdot \frac{1}{W} \cdot \frac{1}{\sqrt{R}}. \tag{30}$$

Evidently, we are able to control the system performance by adjusting the bandwidth W and/or the SNR. Therefore, the TOA based method can perform well for long-range positioning. Figure 1 plots the lower bound of $\sqrt{\mathrm{var}(\hat{d})}$ in Eq. (30) vs. the SNR, for various chip rates ranging from 2Mcps (the top curve) to 8Mcps (the bottom curve).

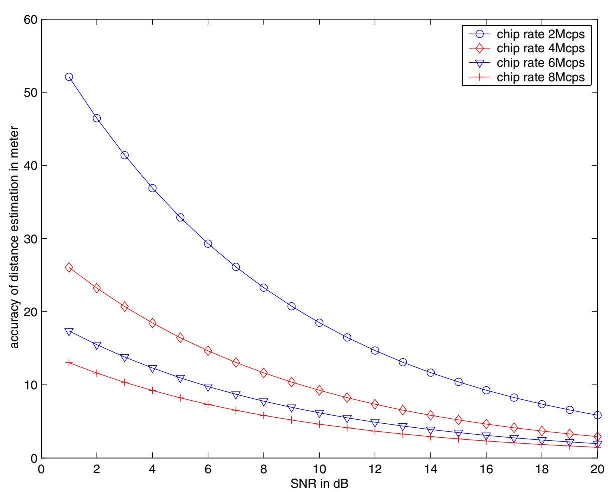

Fig. 1. The distance estimation accuracy of Eq. (30) vs. the SNR in TOA positioning.

By now we have seen that the utility of the SS based method is limited to short-range positioning, while the TOA based method can be used in a wider area. For a quantitative comparison of the "functioning ranges" of these two methods, we introduce the concept of "critical distance" by equating the lower bounds of Eqs. (29) and (26):

$$d_c = \frac{10c}{(2\sqrt{2}\ln 10)\pi} \cdot \frac{\epsilon}{\eta} \cdot \frac{1}{W\sqrt{R}}. \tag{31}$$

Once the signal bandwidth W and the channel characteristics in terms of ϵ, η and SNR are specified, the critical distance d_c can be calculated. As a numerical example, we set $W =$ 5 Mcps, $\epsilon = 4$, $\eta = 8$, and $R =$ 0dB. It follows that d_c is around 25m. Then we are able to predict that the TOA method should outperform the SS method in the range of $d > d_c$, and vice versa. On the other hand, we may use both TOA and SS data to improve a distance estimate. The corresponding CRLB is derived as

$$\begin{aligned}(\mathbf{J}_d)^{-1}_{TOA+SS} &= \frac{1}{(\mathbf{J}_d)_{TOA} + (\mathbf{J}_d)_{SS}} \\ &= \frac{1}{\frac{8\pi^2\beta^2 \cdot R}{c^2} + \left(\frac{10}{\ln 10}\right)^2 \cdot \frac{\epsilon^2}{\eta^2} \cdot \frac{1}{d^2}},\end{aligned} \tag{32}$$

where we assume that the errors in the distance estimates from TOA and SS data are independent. It is straightforward to see

$$(\mathbf{J}_d)^{-1}_{TOA+SS} < \min\left\{(\mathbf{J}_d)^{-1}_{TOA}, (\mathbf{J}_d)^{-1}_{SS}\right\}, \tag{33}$$

where $\min\{a, b\}$ stands for the smaller value of a and b. That is to say, the distance estimation using both TOA and SS data can achieve higher accuracy than the estimation based on only one type of data. However, the improvement is not significant when $(\mathbf{J}_d)_{TOA} << (\mathbf{J}_d)_{SS}$ and $(\mathbf{J}_d)_{TOA} >> (\mathbf{J}_d)_{SS}$, which correspond to $d << d_c$ and $d >> d_c$, respectively.

V. A HYBRID DISTANCE ESTIMATION SCHEME

The above observations lead us to devise a hybrid distance estimation scheme, provided both TOA and SS data are available. Denote $\tilde{d}$ a rough estimate of d, e.g., based on some prior information. The scheme consists of three modes:

- The signal-strength mode. If any prior information suggests $\tilde{d} << d_c$, the SS measurements are the principal data to be employed, because the inclusion of TOA data will not make much improvement for the positioning accuracy.
- The hybrid mode. When $\tilde{d}$ is comparable with d_c, both TOA and SS data should be taken.
- The time-delay mode. The use of TOAs should be dominant for those remote BSs, i.e., when $\tilde{d} >> d_c$.

For the sake of clarity, we express the distance estimate from each of the three modes in a unified formula:

$$\hat{d} = d + \zeta, \tag{34}$$

where ζ is a estimation error, represented by a Gaussian variable $\mathcal{N}(0, \omega^2)$ with

$$\omega^2 = \begin{cases} \frac{c^2}{8\pi^2} \cdot \frac{1}{\beta^2 \cdot R}, & \text{for } \tilde{d} >> d_c, \\ \left(\frac{\ln 10}{10}\right)^2 \cdot \frac{\eta^2}{\epsilon^2} \cdot d^2, & \text{for } \tilde{d} << d_c, \\ \frac{1}{\frac{8\pi^2\beta^2 \cdot R}{c^2} + \left(\frac{10}{\ln 10}\right)^2 \cdot \frac{\epsilon^2}{\eta^2} \cdot \frac{1}{d^2}}, & \text{for } \tilde{d} \sim d_c. \end{cases} \tag{35}$$

Switching among these three modes can be made automatically, depending on the mobile's location vis-a-vis a given BS.

VI. Conclusions

In this paper, we investigated the relations among TOA, TDOA and signal strength based positioning methods. We showed that TDOA positioning may attain the same accuracy as the TOA method under a certain condition, although it cannot do better. By examining the tradeoff between TOA and SS positioning methods, we propose an hybrid distance scheme that uses both TOA and SS data.

Appendix 1

Here are two Examples for $\mathbf{J}_{TDOA}^{-1} \leq \mathbf{J}_{TOA}^{-1}(l)$:

We first provide an example where such an l exists. Consider L BSs, $L > 6$, include three pair of BSs. Each pair of BSs, say $(\mathrm{BS}_i, \mathrm{BS}_{i+3})$, for $i = 1, 2, 3$, is deployed in such a way that the MS is located at the central point of the straight line connecting the two BS locations, which corresponds to

$$w_i = w_{i+3}, \quad \text{and} \quad \boldsymbol{h}_i = -\boldsymbol{h}_{i+3}.$$

By using the relation

$$\sum_{b=1}^{L} w_b \boldsymbol{h}_b \boldsymbol{h}_b^T \geq (\sum_{b=1}^{L} w_b \boldsymbol{h}_b)(\sum_{b=1}^{L} w_b \boldsymbol{h}_b)^T,$$

which is an immediate result from

$$E\left\{(\boldsymbol{h} - \bar{\boldsymbol{h}}) \cdot (\boldsymbol{h} - \bar{\boldsymbol{h}})^T\right\} \geq 0,$$

we have

$$\sum_{b=7}^{L} w_b \boldsymbol{h}_b \boldsymbol{h}_b^T \geq (\sum_{b=7}^{L} w_b \boldsymbol{h}_b)(\sum_{b=7}^{L} w_b \boldsymbol{h}_b)^T.$$

Combining the above equation and

$$\sum_{b=4}^{6} w_b \boldsymbol{h}_b \boldsymbol{h}_b^T \geq (\sum_{b=1}^{6} w_b \boldsymbol{h}_b)(\sum_{b=1}^{6} w_b \boldsymbol{h}_b)^T = \mathbf{0},$$

we obtain

$$\sum_{b=4}^{L} w_b \boldsymbol{h}_b \boldsymbol{h}_b^T \geq (\sum_{b=1}^{L} w_b \boldsymbol{h}_b)(\sum_{b=1}^{L} w_b \boldsymbol{h}_b)^T.$$

With Eq. (23), it is clear that

$$\mathbf{J}_{TOA}^{-1}(L) \leq \mathbf{J}_{TDOA}^{-1} \leq \mathbf{J}_{TOA}^{-1}(3),$$

i.e., $l = 3$ is a choice for this case.

For the second example, consider

$$\sum_{b=1}^{L} w_b \boldsymbol{h}_b \boldsymbol{h}_b^T = (\sum_{b \in \mathcal{L}} w_b \boldsymbol{h}_b)(\sum_{b \in \mathcal{L}} w_b \boldsymbol{h}_b)^T,$$

where

$$\boldsymbol{h}_i = \boldsymbol{h}_j, \quad \text{for all } i, j \in \mathcal{L}.$$

Hence, $l = 0$. It corresponds to the layout where all the BSs and the MS are lined up. Both the TOA and TDOA positioning system collapse in this circumstance, because of the infinite estimation errors.

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